Midterm 1 AM212

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2024-10-26

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1 Problem 1

1.1 Part 1

Consider the BVP

$$\begin{cases} x^2 \frac{d^2 f}{dx^2} + x\frac{df}{dx} = -\lambda f \\ f(1) = 0 \quad f(2) = 0 \end{cases} \tag{1}$$

1.1.1 Show that this is a regular Sturm-Liouville problem and put it in Sturm-Liouville form.

1.1.1.1 Solution

$$x^2 \frac{d^2 f}{dx^2} + x\frac{df}{dx} = -\lambda f \tag{2}$$

$$x\frac{d^2 f}{dx^2} + \frac{df}{dx} = -\frac{1}{x}\lambda f \tag{3}$$

$$\frac{d}{dx}\left(x\frac{df}{dx}\right) = -\lambda \frac{1}{x} f \tag{4}$$

This is an SLP with $w(x) = \frac{1}{x}$ where $w(x)$ is the wieght function.

1.1.2 Find the eigenfunctions and corresponding eigenvalues.

1.1.2.1 Solution

We assume solutions of the form $f(x) = x^\alpha$ where α is complex.

$$f(x) = x^\alpha \tag{5}$$

$$\frac{df}{dx} = \alpha x^{\alpha-1} \tag{6}$$

$$\frac{d^2 f}{dx^2} = \alpha(\alpha-1)x^{\alpha-2} \tag{7}$$

Plugging this into (1) and solving for α we have

$$x^2\alpha(\alpha-1)x^{\alpha-2} + x\alpha x^{\alpha-1} + \lambda x^\alpha = 0 \tag{8}$$

$$\alpha^2 x^\alpha - \alpha x^\alpha + \alpha x^\alpha + \lambda x^\alpha = 0 \tag{9}$$

$$\alpha^2 x^\alpha = -\lambda x^\alpha \tag{10}$$

$$\alpha^2 = -\lambda \tag{11}$$

$$\alpha = \pm i\sqrt{\lambda} \tag{12}$$

The general solution is thus

$$
\begin{aligned}
f(x) &= x^{\pm i\sqrt{\lambda}} && (13)\\
&= e^{\pm i\sqrt{\lambda}\ln(x)} && (14)\\
&= c_1 \cos(\sqrt{\lambda}\ln(x)) + c_2 \sin(\sqrt{\lambda}\ln(x)) && (15)
\end{aligned}
$$

Let us apply the boundary conditions $f(1) = 0$ and $f(2) = 0$

$$
\begin{aligned}
f(1) = 0 &= c_1 \cos(\sqrt{\lambda}\ln(1)) + c_2 \sin(\sqrt{\lambda}\ln(1)) && (16)\\
&= c_1 = 0 && (17)
\end{aligned}
$$

$$
\begin{aligned}
f(2) = 0 = c_2 \sin(\sqrt{\lambda}\ln(2)) && (18)\\
&& (19)
\end{aligned}
$$

In order to satisfy the boundary condition at $x = 2$ we set $\sqrt{\lambda}\ln(2)$ equal to the zeros of the sine function.

$$
\begin{aligned}
\sqrt{\lambda}\ln(2) &= n\pi && (20)\\
\lambda_n &= \left(\frac{n\pi}{\ln(2)}\right)^2 && (21)
\end{aligned}
$$

Thus the eigen-function, eigen-value pair is

$$
f_n(x) = \sin\left(\frac{n\pi\ln(x)}{\ln(2)}\right) \quad \lambda_n = \left(\frac{n\pi}{\ln(2)}\right)^2 \quad n = 1, 2, 3, ... \tag{22}
$$

1.1.3 What is the orthogonality relationship associated with these eigenfunctions?

1.1.3.1 Solution

From Section 1.1.1.1 we know that the weight function $w(x) = \frac{1}{x}$ thus the orthogonality condition is

$$
\langle f_n, f_m \rangle = \int_1^2 \sin\left(\frac{n\pi\ln(x)}{\ln(2)}\right) \sin\left(\frac{m\pi\ln(x)}{\ln(2)}\right) \frac{1}{x} dx = A\delta_{nm} \tag{23}
$$

where A is the normalization constant and

$$\delta_{nm} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Let us find the normalization constant by computing the integral when $n = m$

$$A = \int_1^2 \sin^2\left(\frac{n\pi \ln(x)}{\ln(2)}\right)\frac{1}{x}dx \quad \begin{cases} u & = \ln(x) \\ du & = \frac{1}{x}dx \end{cases} \tag{24}$$

$$= \int_0^{\ln(2)} \sin^2\left(\frac{n\pi u}{\ln(2)}\right)du \tag{25}$$

$$= -\frac{\ln(2)\left(\sin(2\pi n) - 2\pi n\right)}{4\pi n} \quad [1] \tag{26}$$

1.2 Part 2

Consider the equation

$$\begin{cases} x^2\frac{d^2G}{dx^2} + x\frac{dG}{dx} = \delta(x - x') \\ G(1, x') = 0 \quad G(2, x') = 0 \end{cases} \tag{27}$$

1.2.1 Find the Green's function $G(x, x')$ using the method of eigenfunction expansion.

1.2.1.1 Solution

We assume solutions of the following form

$$G(x, x') = \sum_{n=1}^{\infty} c_n(x') \sin\left(\frac{n\pi \ln(x)}{\ln(2)}\right) \tag{28}$$

Let us find the first and second derivative with respect to x.

$$\frac{dG}{dx} = \sum_{\forall n} c_n \frac{n\pi}{\ln(2)}\frac{1}{x}\cos\left(\frac{n\pi \ln(x)}{\ln(2)}\right) \tag{29}$$

$$\frac{d}{dx}\left(\frac{dG}{dx}\right) = \frac{d^2G}{dx^2} = \sum_{\forall n} c_n \frac{n\pi}{\ln(2)}\frac{d}{dx}\left[\frac{1}{x}\cos\left(\frac{n\pi \ln(x)}{\ln(2)}\right)\right] \tag{30}$$

$$= \sum_{\forall n} c_n \frac{n\pi}{\ln(2)}\left(-\frac{1}{x^2}\cos\left(\frac{n\pi \ln(x)}{\ln(2)}\right) - \frac{1}{x^2}\frac{n\pi}{\ln(2)}\sin\left(\frac{n\pi \ln(x)}{\ln(2)}\right)\right) \tag{31}$$

[1]This result is from wolfram alpha

Plugging in to (27), the cosine terms will cancel and we have

$$\sum_{\forall n} -\left(\frac{n\pi}{\ln(2)}\right)^2 c_n \sin\left(\frac{n\pi \ln(x)}{\ln(2)}\right) = \delta(x - x') \tag{32}$$

or in terms of the eigenfunction, eigenvalues is

$$\sum_{\forall n} -\lambda_n c_n(x')\phi_n(x) = \delta(x - x') \tag{33}$$

Let us multiply by $\phi_m(x)$ and the weight function $w(x)$ on both sides and take the integral in [1, 2].

$$\sum_{\forall n} \int_1^2 -\lambda_n c_n(x')\phi_n(x)\phi_m(x)\frac{1}{x}dx = \int_1^2 \delta(x - x')\phi_m(x)\frac{1}{x}dx \tag{34}$$

$$c_m \frac{m^2\pi^2}{\ln(2)^2} \frac{\ln(2)\left(\sin(2\pi m) - 2\pi m\right)}{4\pi m} = \int_1^2 \delta(x - x')\phi_m(x)\frac{1}{x}dx \tag{35}$$

$$c_m(x') = \frac{4\ln(2)}{x' m\pi(\sin(2\pi m) - 2\pi m)} \sin\left(\frac{m\pi \ln(x')}{\ln(2)}\right) \tag{36}$$

the Green's function is thus

$$G(x, x') = \sum_{n=1}^{\infty} \frac{4\ln(2)}{x' n\pi(\sin(2\pi n) - 2\pi n)} \sin\left(\frac{n\pi \ln(x')}{\ln(2)}\right) \sin\left(\frac{n\pi \ln(x)}{\ln(2)}\right) \tag{37}$$

1.2.2 Find the Green's function $G(x, x')$ directly using the 'patching' method

1.2.2.1 Solution

For a singular point x', the Green's function $G(x, x')$ is defined in two parts by

$$\begin{cases} G_L(x, x') & \text{if } x < x' \\ G_R(x, x') & \text{if } x > x' \end{cases} \tag{38}$$

Let us solve the homogeneous problem for equation (27)

$$x^2 \frac{d^2G}{dx^2} + x\frac{dG}{dx} = 0 \tag{39}$$

$$x\frac{d^2G}{dx^2} + \frac{dG}{dx} = 0 \tag{40}$$

$$\frac{d}{dx}\left(x\frac{dG}{dx}\right) = 0 \tag{41}$$

$$x\frac{dG}{dx} = c_1 \tag{42}$$

$$G(x, x') = c_1 \ln|x| + c_2 \tag{43}$$

Applying the left boundary conditions:

$$G_L(1, x') = 0 = c_1 \ln(1) + c_2 \tag{44}$$

$$c_2 = 0 \tag{45}$$

x is bounded in the positive domain so we can drop the absolute value.

$G_L(x, x')$ is thus

$$G_L(x, x') = c_1 \ln(x) \tag{46}$$

Applying the boundary conditions from the right

$$G(2, x') = 0 = k_1 \ln(2) + k_2 \tag{47}$$

$$k_2 = -k_1 \ln(2) \tag{48}$$

$G_R(x, x')$ is thus

$$G_R(x, x') = k_1 \ln(x) - k_1 \ln(2) \tag{49}$$

$$= k_1 \ln\left(\frac{x}{2}\right) \tag{50}$$

at $x = x'$ we enforce continuity

$$G_L(x') = G_R(x') \tag{51}$$

$$c_1 \ln(x') - k_1 \ln\left(\frac{x'}{2}\right) = 0 \tag{52}$$

and the jump condition on the derivative

$$\int_{x'-\epsilon}^{x'+\epsilon} \frac{d}{dx}\left(x\frac{dG}{dx}\right) = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x') \tag{53}$$

$$\left[x\frac{dG}{dx}\right]_{x'-\epsilon}^{x'+\epsilon} = 1 \tag{54}$$

$$\left(x\frac{dG_R}{dx}\right)_{x'+\epsilon} - \left(x\frac{dG_L}{dx}\right)_{x'-\epsilon} = 1 \tag{55}$$

$$(x'+\epsilon)\left(\frac{k_1}{x'+\epsilon}\right) - (x'-\epsilon)\left(\frac{c_1}{x'-\epsilon}\right) = 1 \tag{56}$$

$$-c_1 + k_1 = 1 \tag{57}$$

solving for the system of equations:

$$\begin{bmatrix} \ln(x') & -\ln\left(\frac{x'}{2}\right) \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{58}$$

the determinant of the matrix is $\ln(x') - \ln\left(\frac{x'}{2}\right)$

we have

$$c_1 = \frac{1}{\ln(x') - \ln\left(\frac{x'}{2}\right)} \det\left(\begin{bmatrix} 0 & -\ln\left(\frac{x'}{2}\right) \\ 1 & 1 \end{bmatrix}\right) \tag{59}$$

$$= \frac{\ln\left(\frac{x'}{2}\right)}{\ln(x') - \ln\left(\frac{x'}{2}\right)} \tag{60}$$

$$= \frac{\ln(x') - \ln(2)}{\ln(2)} \tag{61}$$

$$k_1 = \frac{1}{\ln(x') - \ln\left(\frac{x'}{2}\right)} \det\left(\begin{bmatrix} \ln(x') & 0 \\ -1 & 1 \end{bmatrix}\right) \tag{62}$$

$$= \frac{\ln(x')}{\ln(x') - \ln\left(\frac{x'}{2}\right)} \tag{63}$$

$$= \frac{\ln(x')}{\ln(2)} \tag{64}$$

The greens function is thus

$$
\begin{cases} \frac{\ln(x)(\ln(x')-\ln(2))}{\ln(2)} & \text{if } x < x' \\ \frac{\ln(x')(\ln(x)-\ln(2))}{\ln(2)} & \text{if } x > x' \end{cases} \tag{65}
$$

1.2.3 Create a code that plots on the same figure the function $G(x, 5/4)$ using the two different methods. Hand in the code and the figure.

```
using GLMakie

function problem123()
    # figure definition
    fig = Figure()
    ax = Axis(fig[1, 1],
              title = "patching and eigenfunction expansion method",
              xlabel = L"$x$",
              ylabel = L"$G(x, \frac{5}{4})$",
              ylabelrotation = 0)

    # function definition eigenfunction expansion solution
    Gn(x, xp, n) = (
        4*log(2) / (xp*n*π*(sin(2*π*n)-(2*π*n))) *
            sin(n*π*log(xp)/log(2)) *
            sin(n*π*log(x)/log(2))
    )
    # function definition patching method solution
    G(x, xp) = (
        x < xp ?
        log(x)*(log(xp) - log(2)) / log(2) :
        log(xp)*(log(x) - log(2)) / log(2)
    )
    x = LinRange(1.0, 2.0, 1000)
    y = Vector{Float64}(undef, length(x))
    # calculate eigenmodes for each x
    for i in eachindex(x)
        Σ = 0.0
        # keep 1000 eigen modes
        for n in 1:1000
            Σ += Gn(x[i], 5.0/4.0, n)
        end
        y[i] = Σ
    end

    # add the two plots to the figure
```

```
    lines!(ax, x, y, color = :blue, label = L"$\phi$ expansion", linestyle = :dash)
    lines!(ax, x, G.(x, 5.0/4.0), color = :red, label = "patching")
    Legend(fig[1, 2], ax)
    save("problem123.png", fig)
end
problem123();
```

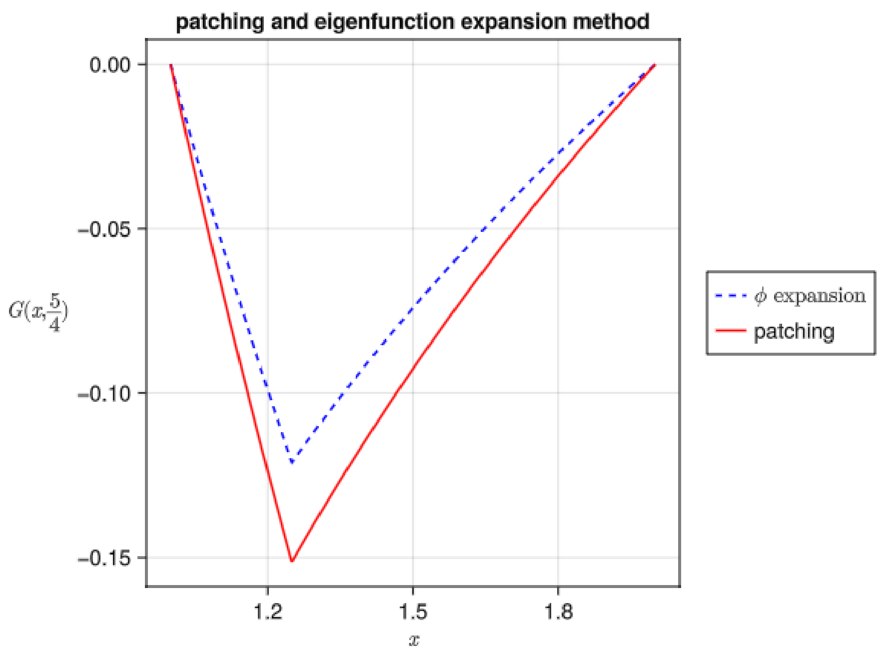

Figure 1: Plot of the Green's function over the domain [1, 2] for the eigen function expansion solution, and the patching method solution

2 Problem 2

Consider the diffusion equation

$$\frac{\partial B}{\partial t} = D\frac{\partial^2 B}{\partial x^2} \tag{66}$$

with boundary conditions and initial condition

$$B(0,t) = F(t) \quad B(L,t) = 0 \tag{67}$$
$$B(x,0) = 0 \tag{68}$$

where $F(t)$ is generic , but you may assume that it satisfies F(0) = 0.

2.0.1 Recast this problem into one that has homogeneous boundary conditions.

2.0.1.1 Solution

We introduce an auxillary function

$$\mu(x,t) = F(t)\left(1 - \frac{x}{L}\right)$$

taking some partial derivatives we have

$$\frac{d\mu}{dt} = \frac{dF}{dt}\left(1 - \frac{x}{L}\right) \tag{69}$$
$$\tag{70}$$
$$\frac{d^2\mu}{dx^2} = 0 \tag{71}$$

we assume the solution has the following form:

$$B(x,t) = v(x,t) + \mu(x,t) \tag{72}$$

plugging into (66)

$$v_t + \mu_t = D\left(v_{xx} + \mu_{xx}\right) \tag{73}$$
$$v_t + \frac{dF}{dt}\left(1 - \frac{x}{L}\right) = Dv_{xx} \tag{74}$$

with boundary and initial conditions

$$
\begin{aligned}
v(0,t) &= B(0,t) - \mu(0,t) && (75)\\
&= F(t) - F(t) && (76)\\
&= 0 && (77)\\
v(L,t) &= B(L,t) - \mu(L,t) && (78)\\
&= 0 && (79)\\
v(x,0) &= B(x,0) - \mu(x,0) && (80)\\
&= 0 - F(0)\left(1 - \frac{x}{L}\right) && (81)\\
&= 0 \quad \text{assuming } F(0) = 0 && (82)
\end{aligned}
$$

The problem thus becomes

$$
\begin{cases}
\frac{dV}{dt} + \frac{dF}{dt}\left(1 - \frac{x}{L}\right) = D\frac{d^2V}{dx^2} \\
v(0,t) = v(L,t) = 0 \\
v(x,0) = 0
\end{cases} \tag{83}
$$

2.0.2 Find the solution for unspecified $F(t)$.

2.0.2.1 Solution

We assume solution of the following form:

$$
v(x,t) = \sum_{\forall n} \psi_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{84}
$$

plugging into the DE we have

$$
\sum_{\forall n} \frac{d\psi_n}{dt} \sin\left(\frac{n\pi x}{L}\right) + \frac{dF}{dt}\left(1 - \frac{x}{L}\right) = -D \sum_{\forall n} \frac{n^2\pi^2}{L^2} \psi_n \sin\left(\frac{n\pi x}{L}\right) \tag{85}
$$

let us multiply by the eigenfunction indexed by m and integrate on both sides from [0, L]

$$\sum_{\forall n} \int_0^L \frac{d\psi_n}{dt} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx + \tag{86}$$

$$\int_0^L \frac{dF}{dt}\left(1 - \frac{x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = -D \sum_{\forall n} \int_0^L \frac{n^2\pi^2}{L^2} \psi_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \tag{87}$$

$$\frac{L}{2}\frac{d\psi_m}{dt} + \frac{dF}{dt}\left(\frac{L(\pi m - \sin(\pi m))}{\pi^2 m^2}\right)^{2} = -\frac{Dm^2\pi^2}{2L}\psi_m \tag{88}$$

$$\frac{d\psi_m}{dt} - D\lambda_m \psi_m = -\frac{dF}{dt}\left(\frac{L(\pi m - \sin(\pi m))}{\pi^2 m^2}\right) \tag{89}$$

Let

$$f_m = -\frac{dF}{dt}\left(\frac{L(\pi m - \sin(\pi m))}{\pi^2 m^2}\right)$$

Let us solve the IVP

$$\frac{d\psi_m}{dt} - D\lambda_m \psi_m = f_m \tag{90}$$

We use method of integrating factor.

Let

$$u(t) = e^{-\int D\lambda_m dt} \tag{91}$$

$$= e^{-D\lambda_m t} \tag{92}$$

multiply by $u(t)$ on both sides and factor

$$\frac{d}{dt}\left(e^{-D\lambda_m t}\psi_m\right) = f_m e^{-D\lambda_m t} \tag{93}$$

$$e^{-D\lambda_m t}\psi_m = \int f_m e^{-D\lambda_m t} dt + c_1 \tag{94}$$

$$\psi_m(t) = e^{D\lambda_m t} \int_0^t f_m(\tau) e^{-D\lambda_m (t-\tau)} d\tau + c_1 \tag{95}$$

applying the initial condition:

[2] the integration is from wolfram alpha

$$\psi_m(0) = 0 = \int_0^0 f_m(0) d\tau + c_1 \tag{96}$$

$$c_1 = 0 \tag{97}$$

The solution to the IVP is thus

$$\psi_m(t) = e^{D\lambda_m t} \int_0^t f_m e^{-D\lambda_m (t-\tau)} d\tau \tag{98}$$

The solution to the full problem is thus

$$B(x,t) = F(t)\left(1 - \frac{x}{L}\right) + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) e^{D\lambda_n t} \int_0^t f_n e^{-D\lambda_n (t-\tau)} d\tau \tag{99}$$

where

$$\lambda_n = -\frac{n^2\pi^2}{L^2} \tag{100}$$

and

$$f_n = -\frac{dF}{dt}\left(\frac{L(\pi n - \sin(\pi n))}{\pi^2 n^2}\right) \tag{101}$$

2.0.3 Create a code that plots the solution $B(x,t)$ for $F(t) = \sin(4\pi t)$, $L = 1$, $D = 1$, at representative time steps. Hand in code and figure.

$$f_n = -4\pi \cos(4\pi t)\left(\frac{L(\pi n - \sin(\pi n))}{\pi^2 n^2}\right) \tag{102}$$

```
function problem203()
    # constants
    L = 1.0
    D = 1.0

    # functions
    F(t) = sin(4*π*t)
    λ(n) = -n^2 * π^2 / L^2
```

```
    fn(n, t) = -4*π * cos(4*π*t) * (L * (π*n - sin(π*n))) / (π^2 * n^2)

    # integral solved using kronrod quadrature
    function integral_term(n, t)
        integrand(τ) = fn(n, τ) * exp(clamp(-D * λ(n) * (t - τ), -700, 700))
        try
            return quadgk(integrand, 0, t, rtol=1e-3, atol=1e-10)[1]
        catch e
            @warn "Integration failed for n=$n, t=$t with error: $e"
            return 0.0  # Return 0 if the integration fails
        end
    end
    function B(x, t)
        sum = 0.0
        # 50 eigen modes
        for n in 1:50
            sum += exp(D*λ(n)*t)*integral_term(n, t)*sin(n*π*x/L)
        end
        return F(t) * (1 - x/L) + sum
    end

    tsteps = [0.0, 0.1, 0.2, 0.3, 0.6]
    x = LinRange(0.0, L, 100)
    fig = Figure()
    ax = Axis(fig[1, 1],
        title = "The solution at representative time steps",
        xlabel = "x",
        ylabel = "B(x, t)",
        ylabelrotation = 0)
    for t in tsteps
        y = [B(xi, t) for xi in x]
        lines!(ax, x, y, label = "t = $t")
    end

    Legend(fig[1, 2], ax)
    save("problem203.png", fig)
end
problem203();
```

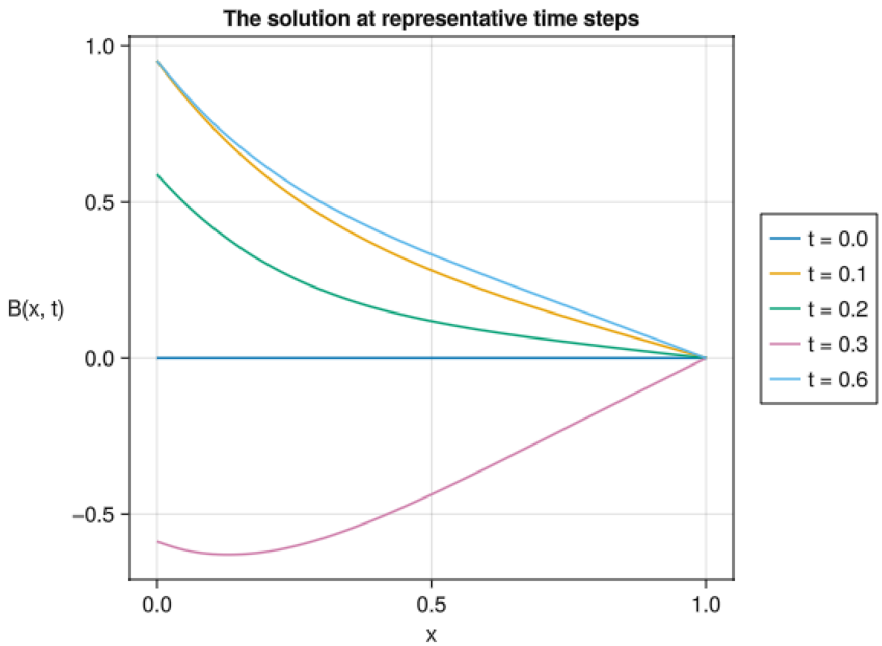

Figure 2: Plot of the solution at representative time steps