Midterm 2 Corrections

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1 Problem 1

For each of the follwing 3 ODEs,

- Plot the numerical solution for $\epsilon \in \{0.1, 0.01, 0.001\}$
- Explain in a few words what method you plan to use to solve this asymptotically and why, based on the numerical solution
- Find the lowest order uniformly convergent analytical approximation to the solution for small positive ϵ
- Compare the numerical and analytical solutions for $\epsilon = 0.01$

1.1 ODE A

$$\begin{cases} \frac{d^2 f}{dt^2} = -f - \epsilon f^2 \left(\frac{df}{dt}\right) \\ f(0) = 1 \quad \frac{df}{dt}(0) = 0 \end{cases} \tag{1}$$

1.1.1 Plot

Let

$$u_1 = f \tag{2}$$
$$u_2 = \frac{df}{dt} \tag{3}$$

$$\dot{u_1} = u_2 \tag{4}$$
$$\dot{u_2} = -u_1 - \epsilon u_2 u_1^2 \tag{5}$$

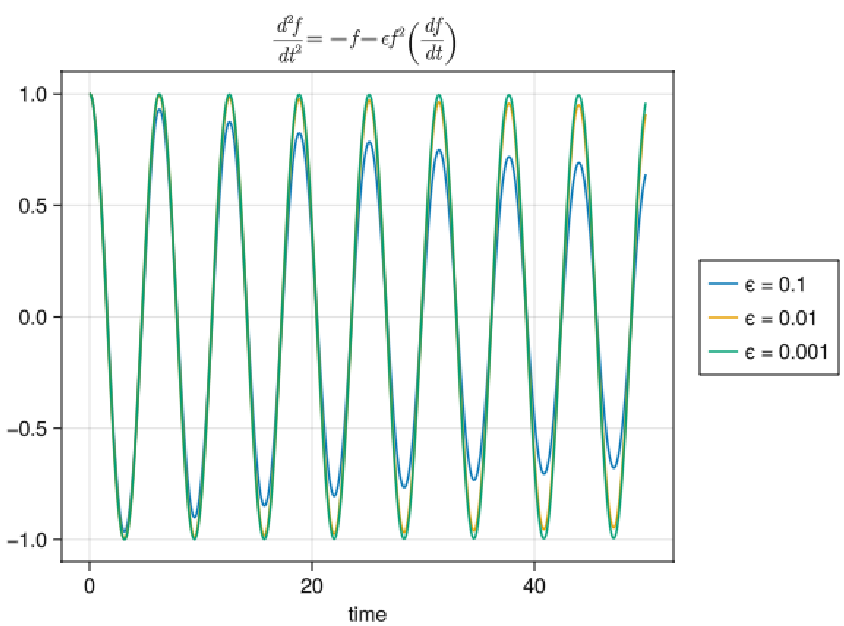

Figure 1: Solution trajectories $t \in [0, 50]$ for $\epsilon \in \{0.1, 0.01, 0.001\}$. See 3.1.1.1 for code.

In order to produce a uniformly convergent approximation we are going to have to employ multiple time scales, the problem oscillates fast and decays slowly in time. Since the equation has no explicit function in time (no $\omega(t)$) terms we can expect method of multiple time scales to work.

1.1.2 Solution

Let

$$\tau_0 = t \tag{6}$$
$$\tau_1 = \epsilon t \tag{7}$$

$$\frac{d\tau_0}{dt} = 1 \tag{8}$$
$$\frac{d\tau_1}{dt} = \epsilon \tag{9}$$

Let us find the first and second derivative operators

$$\frac{d}{dt} = \frac{\partial}{\partial\tau_0} + \epsilon\frac{\partial}{\partial\tau_1} \tag{10}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial\tau_0^2} + 2\epsilon\frac{\partial^2}{\partial\tau_0\tau_1} + \epsilon^2\frac{\partial^2}{\partial\tau_1^2} \tag{11}$$

Let $f(\tau_0, \tau_1) = f_0 + \epsilon f_1$

plugging this into (1) we have

LHS:

$$\left(\frac{\partial^2}{\partial\tau_0^2} + 2\epsilon\frac{\partial^2}{\partial\tau_0\tau_1} + \epsilon^2\frac{\partial^2}{\partial\tau_1^2}\right)(f_0 + \epsilon f_1) \tag{12}$$

RHS:

$$-(f_0 + \epsilon f_1) - \epsilon (f_0 + \epsilon f_1)^2 \left[\left(\frac{\partial}{\partial\tau_0} + \epsilon\frac{\partial}{\partial\tau_1}\right)(f_0 + \epsilon f_1)\right] \tag{13}$$

the lowest order in the expansion is

ϵ^0:

$$\frac{\partial^2 f_0}{\partial\tau_0^2} = -f_0 \tag{14}$$

the solution of which is

$$f_0(\tau_0, \tau_1) = A(\tau_1)e^{i\tau_0} + A^*(\tau_1)e^{-i\tau_0} \tag{15}$$

where A and A^* are complex conjugates

the next order ODE is

$$\frac{\partial^2 f_1}{\partial \tau_0^2} + 2\frac{\partial^2 f_0}{\partial \tau_0 \partial \tau_1} = -f_1 - f_0^2 \frac{\partial f_0}{\partial \tau_0} \tag{16}$$

$$\frac{\partial^2 f_1}{\partial \tau_0^2} + f_1 = -2\frac{\partial^2 f_0}{\partial \tau_0 \partial \tau_1} - f_0^2 \frac{\partial f_0}{\partial \tau_0} \tag{17}$$

Let us plug in our result for f_0 into (17)

RHS:

$$-2\left(i\frac{\partial A}{\partial \tau_1}e^{i\tau_0} - i\frac{\partial A^*}{\partial \tau_1}e^{-i\tau_0}\right) - \left(Ae^{i\tau_0} + A^*e^{-i\tau_0}\right)^2\left(iAe^{i\tau_0} - iA^*e^{-i\tau_0}\right) \tag{18}$$

$$\tag{19}$$

expanding we have

$$-2\left(i\frac{\partial A}{\partial \tau_1}e^{i\tau_0} - i\frac{\partial A^*}{\partial \tau_1}e^{-i\tau_0}\right) - A^3e^{3i\tau_0} - iA^*A^2e^{i\tau_0} + iA^{*2}Ae^{-i\tau_0} + iA^{*3}e^{-3i\tau_0} \tag{20}$$

Notice the equation is symmetric, matching terms in $ie^{i\tau_0}$ and recalling that $AA^* = |A|^2$ let us solve the following ODE:

$$2\frac{\partial A}{\partial \tau_1} = -|A|^2 A \tag{21}$$

Recall $A = |A|e^{i\theta}$

$$2\frac{\partial |A|}{\partial \tau_1} = -|A|^3 \tag{22}$$

$$\int \frac{1}{|A|^3}\partial|A| = -\int \frac{1}{2}\partial \tau_1 \tag{23}$$

$$-\frac{1}{2|A|^2} = -\frac{\tau_1}{2} + c_1 \tag{24}$$

Let us plug in the initial conditions into (15)

$$f_0(0,0) = A(0) + A^*(0) = 1 \tag{25}$$

$$2\Re(A(0)) = 1 \tag{26}$$

$$|A(0)| = \frac{1}{2} \tag{27}$$

$$\frac{\partial f_0}{\partial T_0}(0) = iA(0) - iA^*(0) = 0 \tag{28}$$

$$iA(0) = iA^*(0) \tag{29}$$

(29) implies that A is real valued at time $t = 0$, thus $\theta = 0$

plugging this result into (24) for $\tau_1 = 0$ we have

$$-\frac{1}{2\left(\frac{1}{2}\right)^2} = 0 + c_1 \tag{30}$$

$$c_1 = -2 \tag{31}$$

Solving for A

$$-\frac{1}{2A^2} = -\frac{\tau_1 + 4}{2} \tag{32}$$

$$A^2 = \frac{1}{4 + \tau_1} \tag{33}$$

$$A = \pm\frac{1}{\sqrt{4 + \tau_1}} \tag{34}$$

and

$$f_0(\tau_0, \tau_1) = A(e^{i\tau_0} + e^{-i\tau_0}) \tag{35}$$

$$= 2A(\tau_1)\cos\tau_0 \tag{36}$$

$$\tag{37}$$

finally, it is apparent that $A < 1$ always, thus we can drop the negative term and we have:

$$f_0(t) = \frac{2\cos(t)}{\sqrt{4 + \epsilon t}} \tag{38}$$

1.1.3 Compare

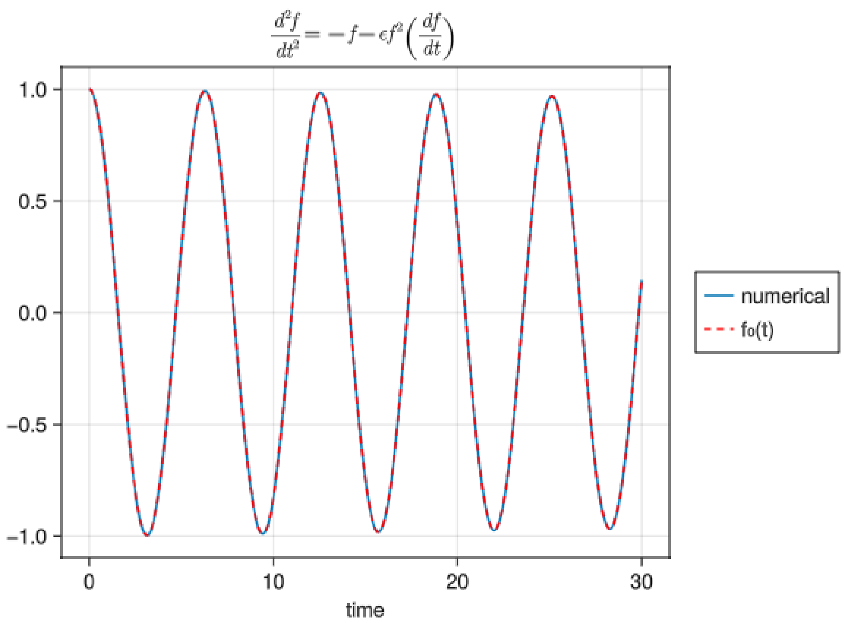

Figure 2: Solution trajectories $t \in [0, 30]$ for $\epsilon = 0.01$. See 3.1.1.1 for code.

1.2 ODE B

$$\begin{cases} \frac{d^2f}{dt^2} = -f + \epsilon f \left(\frac{df}{dt}\right)^4 \\ f(0) = 1 \quad \frac{df}{dt}(0) = 0 \end{cases} \tag{39}$$

1.2.1 Plot

Let

$$u_1 = f \tag{40}$$

$$u_2 = \frac{df}{dt} \tag{41}$$

$$\dot{u_1} = u_2 \tag{42}$$

$$\dot{u_2} = -u_1 + \epsilon u_1 u_2^4 \tag{43}$$

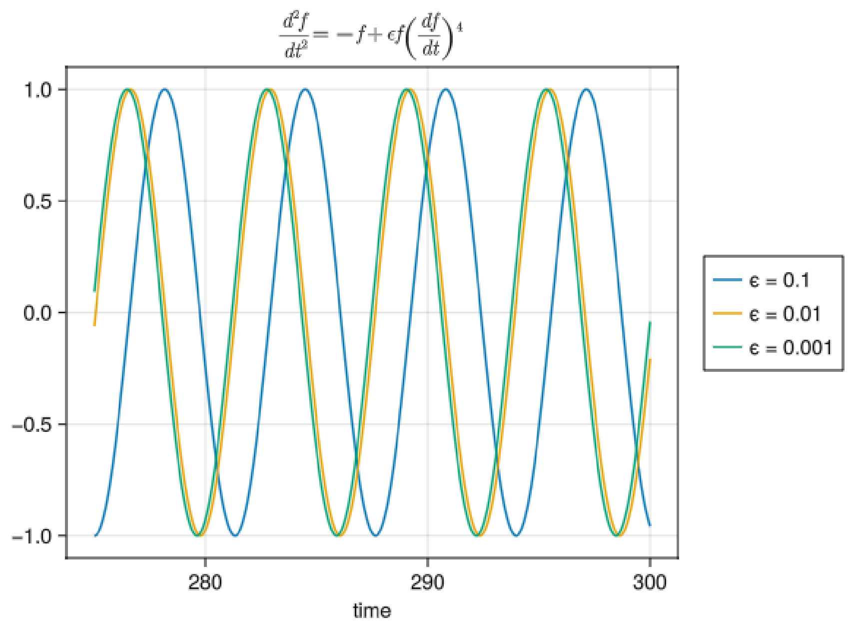

Figure 3: Numerical solution for $\epsilon \in [0.1, 0.01, 0.001]$. See 3.1.1.2 for code.

We can see after numerically solving the ode from $t \in [0, 300]$ and plotting only for $t \in [275, 300]$ that the trajectories do not decay, but the frequency or the period of the motion varies with ϵ as $t \to \infty$. Thus we will use the method of strained coordinates to find a uniform expansion.

1.2.2 Solution

Let

$$f = f_0 + \epsilon f_1 \tag{44}$$
$$\tau = t(1 + a\epsilon) \tag{45}$$

the derivatives of $\frac{df}{dt}$ are

$$\frac{df}{dt} = \frac{df}{d\tau}\frac{d\tau}{dt} = (1 + a\epsilon)\frac{\partial}{\partial \tau}(f_0 + \epsilon f_1) \tag{46}$$
$$\frac{d^2 f}{dt^2} = (1 + a\epsilon)^2 \frac{\partial^2}{\partial \tau^2}(f_0 + \epsilon f_1) \tag{47}$$

Plugging this into (39) we have

$$(1+a\epsilon)^2 \frac{\partial^2}{\partial\tau^2}(f_0+\epsilon f_1) = -(f_0+\epsilon f_1) + \epsilon(f_0+\epsilon f_1)\left[(1+a\epsilon)\frac{\partial}{\partial\tau}(f_0+\epsilon f_1)\right]^4 \tag{48}$$

matching orders of ϵ we have:

O(ϵ^0):

$$\frac{\partial^2 f_0}{\partial\tau^2} = -f_0 \tag{49}$$

with initial conditions $f_0(0) = 1$ and $\frac{df_0}{d\tau}(0) = 0$ we have the particular solution:

$$f_0(\tau) = \cos(\tau) \tag{50}$$

O(ϵ):

$$\begin{aligned}
\frac{\partial^2 f_1}{\partial\tau^2} + f_1 &= -2a\frac{\partial^2 f_0}{\partial\tau^2} + f_0\left(\frac{\partial f_0}{\partial\tau}\right)^4 && (51)\\
&= 2a\cos(\tau) + \cos(\tau)(-\sin(\tau))^4 && (52)\\
&= 2a\cos(\tau) + \frac{1}{16}(\cos(5\tau) - 3\cos(3\tau) + 2\cos(\tau)) && (53)\\
&= \left(2a + \frac{1}{8}\right)\cos(\tau) - \frac{3}{16}\cos(3\tau) + \frac{1}{16}\cos(5\tau) && (54)
\end{aligned}$$

in order to eliminate the source of non-uniformity, we pick an a such that terms in $\cos(\tau)$ are eliminated.

$$\begin{aligned}
2a + \frac{1}{8} &= 0 && (55)\\
a &= -\frac{1}{16} && (56)
\end{aligned}$$

thus we have found that

$$\tau = \left(1 - \frac{\epsilon}{16}\right)t \tag{57}$$

and the uniform expansion is thus:

$$f_0(t) = \cos\left[\left(1 - \frac{\epsilon}{16}\right)t\right] \tag{58}$$

1.2.3 Compare

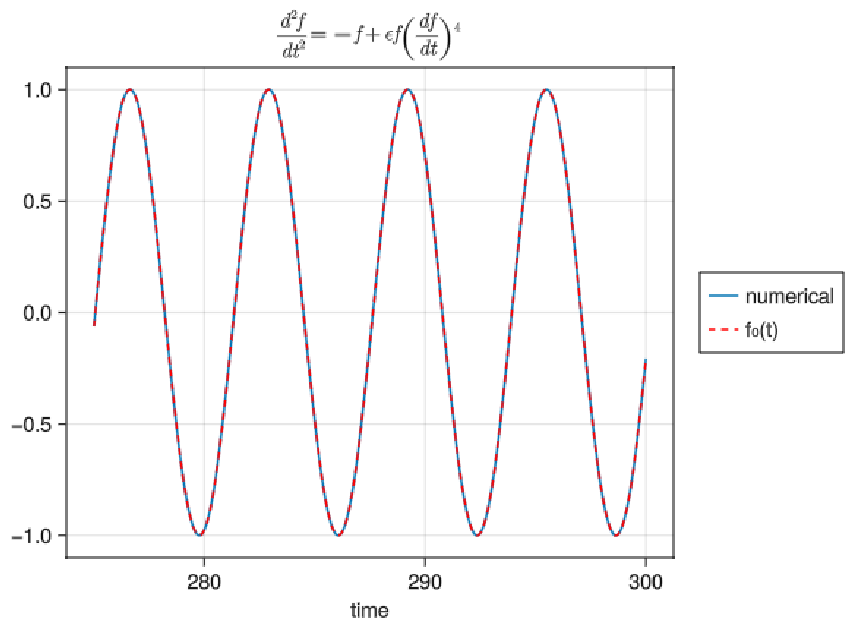

Figure 4: First order approximation of the solution in red-dashed plotted over the numerical solution in blue for $\epsilon = 0.01$. See 3.1.1.2 for code.

Here just as in Figure 3 we plot the numerical solution with $\epsilon = 0.01$ and our uniform expansion for $t \in [275, 300]$ to show that the solution found is correct.

1.3 ODE C

$$\begin{cases} \epsilon \frac{d^2 f}{dt^2} + \frac{df}{dt} + (t+1)\, f = 0 \\ f(0) = 1 \quad f(1) = 2 \end{cases} \tag{59}$$

1.3.1 Plot

Let

$$u_1 = f \tag{60}$$

$$u_2 = \frac{df}{dt} \tag{61}$$

$$\dot{u_1} = u_2 \tag{62}$$

$$\dot{u_2} = -\frac{1}{\epsilon}\left(u_2 + (t+1)u_1\right) \tag{63}$$

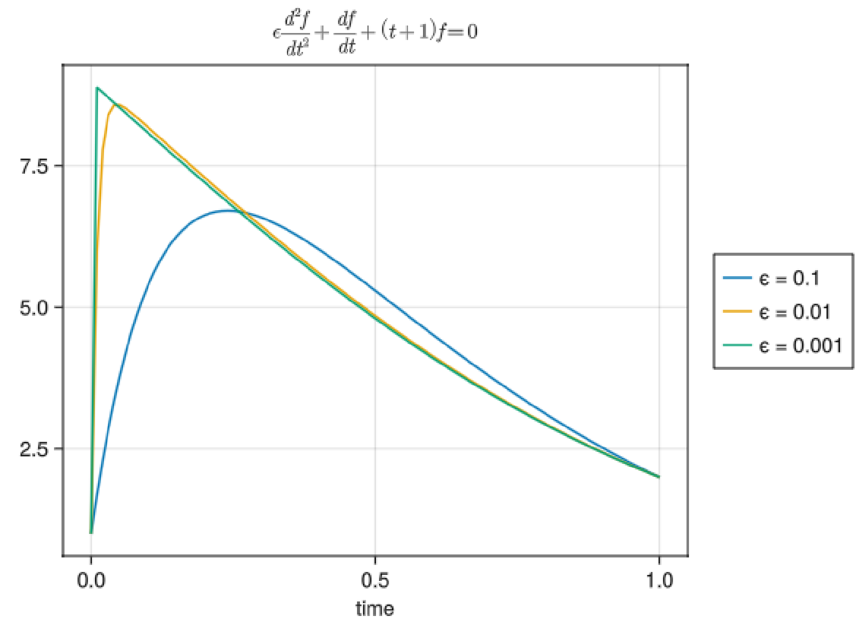

Figure 5: The Solution of the BVP at $\epsilon \in \{0.1, 0.01, 0.001\}$. See 3.1.1.3 for code.

We find the solution using Boundary Value Theory, noticing that the boundary layer is at $t = 0$

1.3.2 Solution

Solve for (59) when $\epsilon = 0$

$$\frac{df_{out}}{dt} + (t+1)f_{out} = 0 \tag{64}$$

$$\tag{65}$$

we solve by method of integrating factor.

$$\mu(t) = e^{\int (t+1)dt} \tag{66}$$

$$= e^{\frac{t^2}{2}+t} \tag{67}$$

multiplying through by integrating factor,

$$\implies \frac{d}{dt}\left(e^{\frac{t^2}{2}+t} f_{out}(t)\right) = 0 \tag{68}$$

$$e^{\frac{t^2}{2}+t} f_{out}(t) = c_1 \tag{69}$$

$$f_{out}(t) = c_1 e^{-\frac{t}{2}(t+2)} \tag{70}$$

applying boundary condition opposite of boundary layer,

$$f_{out}(1) = 2 = c_1 e^{-\frac{1}{2}(1+2)} \tag{71}$$

$$c_1 = 2e^{\frac{3}{2}} \tag{72}$$

thus the solution outside of the BL is

$$f_{out}(t) = 2e^{\frac{3}{2}} e^{-\frac{t}{2}(t+2)} \tag{73}$$

$$\tag{74}$$

To find the inner solution we rescale the t coordinate.

Let $s = \frac{t}{\epsilon^\alpha}$ such that $t = s\epsilon^\alpha$ where $\alpha > 0$, and $s = O(1)$ when $t = O(\epsilon^\alpha)$ within the boundary layer.

this leaves us with the equation

$$\epsilon^{1-2\alpha} \frac{d^2 f_{in}}{ds^2} + \epsilon^{-\alpha} \frac{df_{in}}{ds} + (s\epsilon^\alpha + 1) f_{in} = 0 \tag{75}$$

Notice here that $s\epsilon^\alpha$ is small compared to 1, and when applying the boundary condition $f_{in}(s = 0) = 1$, $f_{in} = O(1)$ in the boundary layer. Let us neglect the term $s\epsilon^\alpha$ assuming it is much smaller than 1.

we have that

$$\epsilon^{1-2\alpha} \frac{d^2 f_{in}}{ds^2} + \epsilon^{-\alpha} \frac{df_{in}}{ds} \simeq -f_{in} \tag{76}$$

Let us neglect the $-f_{in}$ term and find the dominant balance for the following:

$$\epsilon^{1-2\alpha} \frac{d^2 f_{in}}{ds^2} \simeq -\epsilon^{-\alpha} \frac{df_{in}}{ds} \tag{77}$$

we set powers of epsilon equal to eachother and solve for α

$$1 - 2\alpha = -\alpha \tag{78}$$

$$\alpha = 1 \tag{79}$$

plugging in alpha into (77) we solve the following ode:

$$\frac{1}{\epsilon}\frac{d^2 f_{in}}{ds^2} + \frac{1}{\epsilon}\frac{df_{in}}{ds} \simeq 0 \tag{80}$$

$$\frac{d^2 f_{in}}{ds^2} + \frac{df_{in}}{ds} \simeq \epsilon \simeq 0 \tag{81}$$

the general solution of which is:

$$f_{in}(s) = c_1 + c_2 e^{-s} \tag{82}$$

we plug in the boundary condition at the boundary layer,

$$f_{in}(0) = 1 = c_1 + c_2 \tag{83}$$

using Prandtle's matching condition,

$$\lim_{s\to\infty} (c_1 + c_2 e^{-s}) = \lim_{t\to 0} \left(2e^{\frac{3}{2}} e^{-\frac{t}{2}(t+2)}\right) \tag{84}$$

$$c_1 = 2e^{\frac{3}{2}} \tag{85}$$

where c_1 represents the common limit of Prandtl's matching condition which we will call L

$$1 = 2e^{\frac{3}{2}} + c_2 \tag{86}$$

$$c_2 = \left(1 - 2e^{\frac{3}{2}}\right) \tag{87}$$

we can now assemble the solution for $f(t)$

Recall that $s = \frac{t}{\epsilon}$

$$f(t) = f_{in}(t) + f_{out}(t) - L \tag{88}$$

$$= \cancel{2e^{\frac{3}{2}}} + \left(1 - 2e^{\frac{3}{2}}\right) e^{-\frac{t}{\epsilon}} + 2e^{\frac{3}{2}} e^{-\frac{t}{2}(t+2)} - \cancel{2e^{\frac{3}{2}}} \tag{89}$$

$$= \left(1 - 2e^{\frac{3}{2}}\right) e^{-\frac{t}{\epsilon}} + 2e^{\frac{3}{2}} e^{-\frac{t}{2}(t+2)} \tag{90}$$

1.3.3 Compare

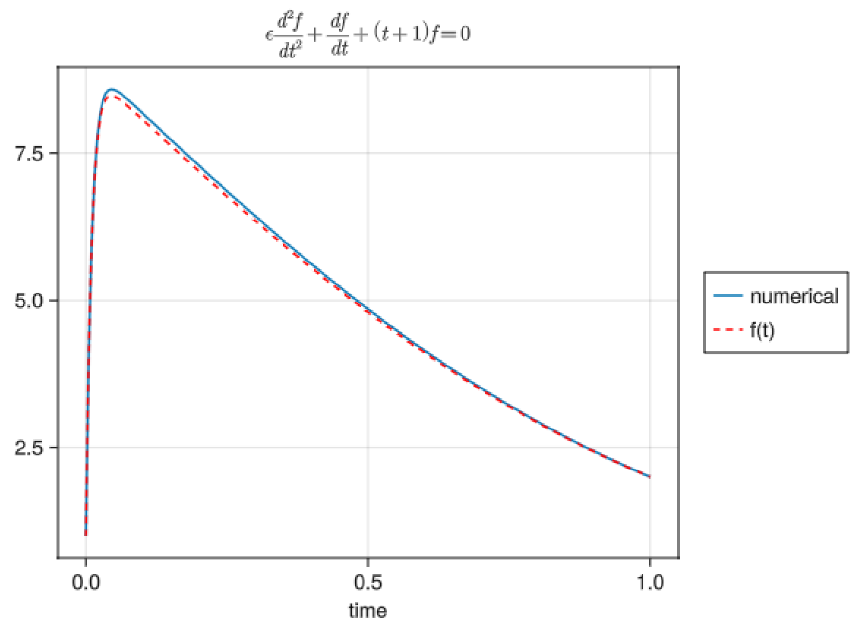

Figure 6: Numerical vs analytical solution of the ODE C Problem $\epsilon = 0.01$. See 3.1.1.3 for code.

2 Problem 2

Find the eigenvalues and eigenfunctions of this eigenvalue problem, in the limit where the eigenvalue λ is very large and positive.

$$\begin{cases} \frac{d^2 f}{dx^2} + \lambda \left(x+1\right)^2 f = 0 \\ f(1) = 0 \quad f(2) = 0 \end{cases} \tag{91}$$

2.1 Solution

We solve the problem using WKB theory assuming our large eigenvalue is a function of a small parameter $\epsilon << 1$.

Let

$$\lambda = \frac{1}{\epsilon^2} \tag{92}$$

$$x_s = x \tag{93}$$

$$x_f = \frac{g(x_s)}{\epsilon} \tag{94}$$

$$f = f_0 + \epsilon f_1 \tag{95}$$

Let us find the derivative operators

$$\frac{\partial x_s}{\partial x} = 1 \tag{96}$$

$$\frac{\partial x_f}{\partial x} = \frac{g'(x_s)}{\epsilon} \tag{97}$$

$$\begin{aligned}
\frac{d}{dx} &= \frac{\partial}{\partial x_s}\frac{\partial x_s}{\partial x} + \frac{\partial}{\partial x_f}\frac{\partial x_f}{\partial dx} && (98)\\
&= \frac{\partial}{\partial x_s} + \frac{g'(x_s)}{\epsilon}\frac{\partial}{\partial x_f} && (99)
\end{aligned}$$

$$\begin{aligned}
\frac{d^2}{dx^2} &= \frac{\partial}{\partial x_s}\left(\frac{\partial}{\partial x_s} + \frac{g'(x_s)}{\epsilon}\frac{\partial}{\partial x_f}\right) + \frac{g'(x_s)}{\epsilon}\frac{\partial}{\partial x_f}\left(\frac{\partial}{\partial x_s} + \frac{g'(x_s)}{\epsilon}\frac{\partial}{\partial x_f}\right) && (100)\\
&= \frac{\partial^2}{\partial x_s^2} + \frac{g''(x_s)}{\epsilon}\frac{\partial}{\partial x_f} + \frac{2g'(x_s)}{\epsilon}\frac{\partial^2}{\partial x_s \partial x_f} + \frac{g'(x_s)^2}{\epsilon^2}\frac{\partial^2}{\partial x_f^2} && (101)
\end{aligned}$$

Plugging this into (91) we have

$$\left(\frac{\partial^2}{\partial x_s^2} + \frac{g''(x_s)}{\epsilon}\frac{\partial}{\partial x_f} + \frac{2g'(x_s)}{\epsilon}\frac{\partial^2}{\partial x_s \partial x_f} + \frac{g'(x_s)^2}{\epsilon^2}\frac{\partial^2}{\partial x_f^2}\right)(f_0 + \epsilon f_1) + \frac{1}{\epsilon^2}(x+1)^2(f_0 + \epsilon f_1) = 0 \tag{102}$$

equating terms in order $O(\epsilon^{-2})$

$$g'(x_s)^2\frac{\partial^2 f_0}{\partial x_f^2} + (x+1)^2 f_0 = 0 \tag{103}$$

In order to eliminate secular terms we have:

$$\begin{aligned}
g'^2(x_s) &= (x+1)^2 && (104)\\
g'(x_s) &= (x+1) \quad \text{where} \quad x > 0 \quad \forall x && (105)\\
g''(x_s) &= 1 && (106)\\
g(x_s) &= \int_1^x z + 1dz && (107)\\
&= \frac{1}{2}\left(x^2 + 2x - 3\right) && (108)
\end{aligned}$$

the general solution to the PDE (103) is thus

$$f_0(x_s, x_f) = A(x_s)\cos(x_f) + B(x_s)\sin(x_f) \tag{109}$$

Let us find the coefficients $A(x_s), B(x_s)$ by matching terms in the next order $O(\epsilon^{-1})$

$$g''(x_s)\frac{\partial f_0}{\partial x_f} + 2g'(x_s)\frac{\partial^2 f_0}{\partial x_s \partial x_f} + g'^2(x_s)\frac{\partial^2 f_1}{\partial x_f^2} + (x_s+1)^2 f_1 = 0 \tag{110}$$

$$g'^2(x_s)\frac{\partial^2 f_1}{\partial x_f^2} + (x_s+1)^2 f_1 = -g''(x_s)\frac{\partial f_0}{\partial x_f} - 2g'(x_s)\frac{\partial^2 f_0}{\partial x_s \partial x_f} \tag{111}$$

plugging in the partials of (109) into the RHS:

$$-g''(B\cos(x_f) - A\sin(x_f)) - 2g'\left(\frac{\partial B}{\partial x_s}\cos(x_f) - \frac{\partial A}{\partial x_s}\right) \tag{112}$$

equating terms in $\cos(x_f)$ we have:

$$-B - 2(x_s+1)\frac{dB}{dx_s} = 0 \tag{113}$$

$$\frac{dB}{B} = -\frac{1}{2(x_s+1)} \tag{114}$$

$$\int \frac{1}{B} dB = -\frac{1}{2}\int \frac{1}{x_s+1} dx_s \tag{115}$$

$$\ln(B) = -\frac{1}{2}\ln(x_s+1) + c_1 \tag{116}$$

$$B(x_s) = \frac{c_1}{\sqrt{x_s+1}} \tag{117}$$

since $B(x_s), A(x_s)$ are the same equation we can write out the first order approximation as:

$$f_0(x_s, x_f) = \frac{c_1}{\sqrt{x_s+1}}\cos(x_f) + \frac{c_2}{\sqrt{x_s+1}}\sin(x_f) \tag{118}$$

Recall that (93) and (94)

$$f_0(x) = \frac{c_1}{\sqrt{x+1}}\cos\left(\frac{\frac{1}{2}\left(x^2+2x-3\right)}{\epsilon}\right) + \frac{c_2}{\sqrt{x+1}}\sin\left(\frac{\frac{1}{2}\left(x^2+2x-3\right)}{\epsilon}\right) \tag{119}$$

Plugging in boundary conditions we have:

$$f_0(1) = \frac{c_1}{\sqrt{2}} = 0 \tag{120}$$
$$c_1 = 0 \tag{121}$$

$$f_0(2) = \frac{c_2}{\sqrt{3}} \sin\left(\frac{5}{2\epsilon}\right) = 0 \tag{122}$$
$$\tag{123}$$

solving epsilon for the zeros of the sine function

$$\frac{5}{2\epsilon} = n\pi \tag{124}$$
$$\epsilon = \frac{5}{2n\pi} \tag{125}$$
$$\tag{126}$$

the coefficient c_2 is an undetermined coefficient.

the eigen function and eigenvalues are:

$$\phi_n(x) = \sin\left(\frac{\pi n(x^2 + 2x - 3}{5}\right) \quad \lambda_n = \frac{1}{\epsilon^2} = \left(\frac{2n\pi}{5}\right)^2 \quad \text{for} \quad n = 1, 2, 3, ... \tag{127}$$

the first order approximation of the BVP is

$$f_0(x) = \frac{A_n}{\sqrt{x+1}}\phi_n(x) \tag{128}$$

3 Appendix

3.1 Code

3.1.1 Problem 1

3.1.1.1 Part A

```
using GLMakie
using DifferentialEquations
using StaticArrays

ode1(u, p, t) = SVector{2}(u[2], -u[1] -p*u[2]*(u[1]^2.0))
f0p(t, ϵ) = 2.0*cos(t) / sqrt(4.0 +ϵ*t)
function p1pa()
    # tspan, initial conditions, params
    ϵ = SVector{3}(0.1, 0.01, 0.001)
    t0 = 0.0
    tf = 50.0
    u0 = SVector{2}(1, 0)
    tspan = LinRange(t0, tf, 1000)
    # figure stuff
    fig = Figure()
    ax = Axis(fig[1, 1],
        title = L"$\frac{d^2f}{dt^2} = -f -\epsilon f^2 \left(\frac{df}{dt}\right)$",
        xlabel = "time"
    )

    for p in ϵ
        # solve ode
        sol = solve(ODEProblem(ode1, u0, (t0, tf), p),
            Tsit5(), saveat=0.01, abstol= 1e-8, reltol = 1e-8)
        # plot solution
        lines!(ax, sol.t, sol[1, :], label = "ϵ = $p")
    end
    lines!(ax, tspan, f0p.(tspan, 0.01), color = :red, linestyle = :dash)
    Legend(fig[1, 2], ax)
    display(fig)
    #save("p1pa.png", fig)
end
function p1pd()
    # tspan, initial conditions, params
    ϵ = 0.01
    t0 = 0.0
    tf = 30.0
    u0 = SVector{2}(1, 0)
    tspan = LinRange(t0, tf, 1000)
    # figure stuff
    fig = Figure()
    ax = Axis(fig[1, 1],
        title = L"$\frac{d^2f}{dt^2} = -f -\epsilon f^2 \left(\frac{df}{dt}\right)$",
        xlabel = "time"
    )
    # solve ode
```

```
    sol = solve(ODEProblem(ode1, u0, (t0, tf), ϵ),
        Tsit5(), saveat=0.01, abstol= 1e-8, reltol = 1e-8)
    # plot solution
    lines!(ax, sol.t, sol[1, :], label = "numerical")
    lines!(ax, tspan, f0p.(tspan, ϵ), color = :red, linestyle = :dash, label = "f₀(t)")
    Legend(fig[1, 2], ax)
    #display(fig)
    save("p1pd.png", fig)
end
```

3.1.1.2 Part B

```
using GLMakie
using DifferentialEquations
using StaticArrays

ode2(u, p, t) = SVector{2}(u[2], -u[1] + p * u[1] * u[2]^4.0)
function p2pa()
    # tspan, initial conditions, params
    ϵ = SVector{3}(0.1, 0.01, 0.001)
    t0 = 0.0
    tf = 300.0
    u0 = SVector{2}(1, 0)
    # figure stuff
    fig = Figure()
    ax = Axis(fig[1, 1],
        title = L"$\frac{d^2 f}{dt^2} = -f + \epsilon f \left(\frac{df}{dt}\right)^4$",
        xlabel = "time"
    )
    for p in ϵ
        sol = solve(ODEProblem(ode2, u0, (t0, tf), p),
            Tsit5(), saveat=0.01, abstol= 1e-8, reltol = 1e-8)
        idx = findall(t -> t >= 275.0, sol.t)

        lines!(ax, sol.t[idx], sol[1, idx], label = "ϵ = $p")
    end
    Legend(fig[1, 2], ax)
    #display(fig)
    save("p2pa.png", fig)
end
f₀(t, ϵ) = cos((1.0 - (ϵ)/16.0)*t)
function p2pd()
    # tspan, initial conditions, params
    ϵ = 0.01
    t0 = 0.0
    tf = 300.0
```

```
    tspan = LinRange(275.0, tf, 10000)
    u0 = SVector{2}(1, 0)
    # figure stuff
    fig = Figure()
    ax = Axis(fig[1, 1],
        title = L"$\frac{d^2 f}{dt^2} = -f + \epsilon f \left(\frac{df}{dt}\right)^4$",
        xlabel = "time"
    )
    sol = solve(ODEProblem(ode2, u0, (t0, tf), ϵ),
        Tsit5(), saveat=0.01, abstol= 1e-8, reltol = 1e-8)
    idx = findall(t -> t >= 275.0, sol.t)
    lines!(ax, sol.t[idx], sol[1,idx], label = "numerical")
    lines!(ax, tspan, f₀.(tspan, ϵ), linestyle = :dash, label = "f₀(t)", color = :red)
    Legend(fig[1, 2], ax)
    #display(fig)
    save("p2pd.png", fig)
end
```

3.1.1.3 Part C

```
using GLMakie
using DifferentialEquations
using StaticArrays
using BoundaryValueDiffEq

function odec!(du, u, p, t)
    du[1] = u[2]
    du[2] = -1.0/p * (u[2] + (t + 1)*u[1])
end

function bc!(res, u, p, t)
    res[1] = u[1][1] - 1.0
    res[2] = u[end][1] - 2.0
end

function init_guess(x)
    [1.0 + x, 0.0]
end

function f(t, ϵ)
    (1 - 2.0*exp(3.0/2.0))*exp(-t/ϵ) + 2.0*exp(3.0/2.0)*exp(-(t/2.0)*(t+2.0))
end

function p3pa()
    ϵ = SVector{3}(0.1, 0.01, 0.001)
    fig = Figure();display(fig)
```

```
    ax = Axis(fig[1, 1],
        title = L"\epsilon \frac{d^2 f}{dt^2} + \frac{df}{dt} + \left(t + 1\right)f = 0",
    xlabel = "time")
    for p in ϵ
        prob = BVProblem(odec!, bc!, init_guess, (0.0, 1.0), p)
        sol = solve(prob, Shooting(Vern7()), saveat = 0.01)
        lines!(ax, sol.t, sol[1, :], label = "ϵ = $p")
    end

    Legend(fig[1, 2], ax)
    save("p3pa.png", fig)
end

function p3pd()
    ϵ = 0.01
    fig = Figure()
    tspan = LinRange(0.0, 1.0, 1000)
    ax = Axis(fig[1, 1],
        title = L"\epsilon \frac{d^2 f}{dt^2} + \frac{df}{dt} + \left(t + 1\right)f = 0",
    xlabel = "time")
    prob = BVProblem(odec!, bc!, init_guess, (0.0, 1.0), ϵ)
    sol = solve(prob, Shooting(Vern7()), saveat = 0.005)
    lines!(ax, sol.t, sol[1, :], label = "numerical")
    lines!(ax, tspan, f.(tspan, ϵ), label = "f(t)", linestyle = :dash, color = :red)
    Legend(fig[1, 2], ax)
    display(fig)
    save("p3pd.png", fig)
end
```