AM212 Lectures

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Contents

Chapter 1

Ideas of dimensional analysis

Lecture notes edited by Howard, Henry, Moein, Alexandra and Kevin.
In this set of lectures we will explore a few key analytical tools of applied mathematics. We will review general ideas for the solution of linear PDEs in Chapter 2, dive in Chapter 3 into asymptotic methods for the solution of ODEs (many of which extend fairly easily to PDEs as well), and finally introduce the concept of variational calculus as a method for optimizing functionals in Chapter 4.

Before we begin, however, we will spend this first lecture learning what is arguably *the* biggest 'bang-for-the-buck' tool of applied mathematics, namely dimensional analysis, which will allow you to obtain insightful and interesting results using just back-of-the-envelope calculations.

1.1 The notion of dimensions and units

First of all, let's remember that we are *applied* mathematicians, and that every equation that we will likely have to solve arises from an application in real life. This means that the quantities modelled (parameters, variables, functions) are usually dimensional and only make sense when expressed in some system of units.

A *dimension* refers to the type of the variable: mass, length, time, temperature, etc., or combinations of these dimensions (e.g. velocity has the dimensions of length/time). In these lectures, we will use the notation $[q]$ to denote 'the dimension of q'. For instance, if the variable x is a length and v is a velocity, we would write

$$[x] = \text{ length}, \quad [v] = \frac{\text{length}}{\text{time}} \tag{1.1}$$

There are a few fundamental dimensions: mass, length, time, temperature, electric charge. Most other quantities have dimensions that can be written in terms of these fundamental ones. See the NRL Plasma Formulary for detail.

The *unit* refers to the metric by which this dimension is measured, e.g.:

- centimeters, meters, kilometers, etc. for length
- seconds, minutes, hours, years, etc. for time
- grams, kilograms, etc. for mass
- degrees (usually Kelvin) for temperature
- Coulomb, or Statcoulomb for charge

Note that the international community generally uses one of two standard systems of units: the SI ('Système International' units, meters-kilograms-seconds) or cgs (centimeters-grams-seconds) systems. The NRL Plasma Formulary provides a table to convert quantities between different unit systems.

But as we shall discuss in this lecture, it may sometimes be much better to use other systems of units that are more appropriate to the problem we want to solve[1].

The presence of dimensions and units in real-life problems has a few trivial but nonetheless useful / important consequences. For instance, the left-hand side and right-hand side of any equation describing

[1] There is an important exception: scientists should always refrain from using inches, feet, ounces, and other relics from the imperial unit system, otherwise bad things happen — Google "Mars Climate Orbiter" for example.

that problem *must* have the same units – otherwise we are comparing apples to oranges. This is a simple way of checking whether your equations are correct or not. It can also be used to *find* the dimensions of certain parameters in an equation, and use that information to model them if needed.

Example 1: Why is it $E = mc^2$ and not $E = mc^3$ or $E = m^2c^2$?

Solution: Energy has units of $\frac{\text{mass}\cdot\text{length}^2}{\text{time}^2}$. By analyzing the dimensions of the right-hand side of the equation, we get that m has dimensions of mass and c, the speed of light, has dimensions of $\frac{\text{length}}{\text{time}}$. This gives matching dimensions for both sides of the equation. However, $E = mc^3$ and $E = m^2c^2$ give incorrect right-hand side dimensions of $\frac{\text{mass}\cdot\text{length}^3}{\text{time}^3}$ and $\frac{\text{mass}^2\cdot\text{length}^2}{\text{time}^2}$ respectively.

Example 2: In the diffusion equation $\partial f/\partial t = D\partial^2 f/\partial x^2$, what is the dimension of D? In many applications, D is often modeled as a characteristic velocity times a characteristic length scale. For example, in a turbulent flow, D is modeled as the typical length scale of a turbulent eddy, times the velocity of that eddy. In random walk processes, D is modeled as the typical travel velocity times the characteristic length scale traveled before changing direction. Why does that make sense (at least, dimensionally)?

Solution:

$$\frac{[\partial f]}{[\partial t]} \implies \frac{[f]}{\text{time}}$$
$$[D]\frac{[\partial^2 f]}{[\partial x^2]} \implies [D]\frac{[f]}{\text{length}^2}$$
$$\frac{[f]}{\text{time}} = [D]\frac{[f]}{\text{length}^2} \implies [D] = \frac{\text{length}^2}{\text{time}}$$

D will have dimensions of $\frac{\text{length}^2}{\text{time}}$. In both turbulent flows and random walk processes, D is modeled as a velocity times a length. Dimensionally, this makes sense because the product of velocity, which has dimensions of $\frac{\text{length}}{\text{time}}$, and length give the proper dimensions of D.

1.2 Introduction to dimensional analysis

The idea behind dimensional analysis is the following. For a simple problem that only has a few input parameters (each of which has their own dimensions), it is usually possible to deduce what the characteristic length scale(s) or time scale(s) or velocity scale(s) of the problem ought to be, simply by finding the combination of parameters that has the correct dimension. Using that information, we can often learn something interesting and useful about the solution.

Example 1: Consider an object of mass m, oscillating sideways on a spring with tension coefficient k. The time evolution of its displacement from rest x is described by the harmonic oscillator equation:

$$m\frac{d^2x}{dt^2} = -kx \tag{1.2}$$

- What is the dimension of k?
- Using dimensional analysis with the parameters provided, show that there is a single emergent characteristic timescale describing the problem.
- Solve the problem exactly, assuming that the mass starts with zero velocity at a position x_0 away from rest. Does this characteristic time scale indeed appear in the solution?

Solution: If we take x to have dimension of length and then plug the units into the equation we have:

$$[m\frac{d^2x}{dt^2}] = \frac{\text{mass}\cdot\text{length}}{\text{time}^2} = [kx] = [k]\cdot\text{length} \implies [k] = \frac{\text{mass}}{\text{time}^2}$$

In order to get a timescale out of this, we will need to divide out the mass (m) from the dimension of k and then square root in order to remove the exponent on time. This produces the characteristic timescale:

$$\tau = \sqrt{\frac{m}{k}}$$

which indeed has dimension of time.

We solve the second order differential equation and associated conditions:

$$m\frac{d^2x}{dt^2} + kx = 0$$
$$x(0) = x_0$$
$$x'(0) = 0$$

This produces the characteristic polynomial:

$$mr^2 + k = 0$$

which has solutions

$$r = \pm i\sqrt{\frac{k}{m}} = \pm i\frac{1}{\tau}$$

giving the general solution:

$$x(t) = c_1\cos(\frac{t}{\tau}) + c_2\sin(\frac{t}{\tau})$$
$$x'(t) = -\frac{c_1}{\tau}\sin(\frac{t}{\tau}) + \frac{c_2}{\tau}\cos(\frac{t}{\tau})$$

Applying our velocity BC gives:

$$x'(0) = 0 = c_2 \implies x(t) = c_1\cos(\frac{t}{\tau})$$

while the IC gives:

$$x(0) = x_0 = c_1.$$

Giving the final solution:

$$x(t) = x_0\cos(\frac{t}{\tau})$$

We see the characteristic timescale τ appear as expected.

Example 2: Let's now add some damping, so the equation reads

$$m\frac{d^2x}{dt^2} = -kx - \lambda\frac{dx}{dt} \tag{1.3}$$

- Show that there are two distinct characteristic timescales in the problem.
- What is their physical meaning?

Solution: Let us non-dimensionalize equation 1.3.

$$m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = 0$$

Let $\hat{x} = \frac{x}{x_s}$ and $\hat{t} = \frac{t}{t_s}$ and plug in for x and t.

$$\frac{mx_s}{t_s^2}\frac{d^2\hat{x}}{d\hat{t}^2} + \frac{\lambda x_s}{t_s}\frac{d\hat{x}}{d\hat{t}} + k\hat{x}x_s = 0$$
$$\frac{d^2\hat{x}}{d\hat{t}^2} + \frac{\lambda t_s}{m}\frac{d\hat{x}}{d\hat{t}} + \frac{kt_s^2}{m}\hat{x} = 0$$

By setting the leading coefficients in front of $\hat{x}$ and $\frac{d\hat{x}}{d\hat{t}}$ equal to 1 we can find the two characteristic time scales for the mass spring damped oscillator:

$$\frac{\lambda t_s}{m} = 1 \Longrightarrow t_s = \tau_1 = \frac{m}{\lambda}$$
$$\frac{k t_s^2}{m} = 1 \Longrightarrow t_s = \tau_2 = \sqrt{\frac{m}{k}}$$

The first, as we saw above, is the characteristic oscillation timescale in the absence of damping. The second timescale depends on the damping coefficient λ, and we may therefore hypothesize (and can verify) that it is the damping timescale of the oscillator.

Example 3: The advection-diffusion equation for temperature is given by

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = D\frac{\partial^2 T}{\partial x^2} \tag{1.4}$$

where v is a constant advection velocity, and D is a constant diffusion coefficient. We consider this equation on the interval $[0, L]$ where L is a length, such that $T(0) = T_0$ and $T(L) = 2T_0$.

- Show using dimensional analysis that there is a second characteristic length scale of this problem in addition to L.
- Solve for the steady-state solution of this problem. Does this new characteristic length scale appear in the solution?

Solution:
There are two parameters in the equation: v and D which have dimension:

$$[v] = \frac{l}{t}$$
$$[D] = \frac{l^2}{t}$$

We are trying to isolate an l with a combination of v and D. If you think about it, it becomes clear this can be generated from:

$$[\frac{D}{v}] = \frac{\text{length}^2}{\text{time}}\frac{\text{time}}{\text{length}} = \text{length}$$

Giving us:

$$L_c = \frac{D}{v}$$

We now solve the steady state equation:

$$v\frac{dT}{dx} = D\frac{d^2T}{dx^2} \implies \frac{dT}{dx} = \frac{D}{v}\frac{d^2T}{dx^2}$$
$$\implies L_c\frac{d^2T}{dx^2} - \frac{dT}{dx} = 0$$

This has corresponding characteristic polynomial:

$$L_c r^2 - r = 0$$

Which gives the roots:

$$r_{1,2} = 0, 1/L_c$$

Producing the general solution:

$$T = c_1 + c_2 e^{\frac{x}{L_c}}$$

Showing that this characteristic length scale appears in the steady state solution.

1.3 Non-dimensional equations

Having discovered that a given problem has 'natural' length, time, or velocity scales, that are given by the input parameters, the next natural step is to use these characteristic scales as our new metric (our new system of units). By casting each variable (dependent and independent) in the new unit system, we can create a non-dimensional problem (i.e. a problem which has no dimensions).

Example 1: Let's go back to the harmonic oscillator example of the previous section.

- Create new dimensionless variables $\hat{x} = x/x_0$, and $\hat{t} = t/\tau$ where τ is the characteristic timescale of the harmonic oscillator that was derived in the previous section. Express the governing equation and initial conditions in this new set of variables, and show that there are no longer any parameters left – this is a 'universal' problem.
- What is the 'universal' solution $\hat{x}(\hat{t})$? for these initial conditions? Show that it recovers the solution found earlier upon changing back to dimensional variables.

Solution: We first get our original variables in terms of these new dimensionless variables:

$$\hat{x} = x/x_0 \implies x = \hat{x}x_0$$
$$\hat{t} = t/\tau \implies t = \hat{t}\tau$$

We now plug these into our original equation:

$$\begin{aligned} m\frac{d^2x}{dt^2} + kx = 0 &\implies m\frac{d^2(\hat{x}x_0)}{d(\hat{t}\tau)^2} + k(\hat{x}x_0) = 0 \\ &\implies \frac{mx_0}{\tau^2}\frac{d^2\hat{x}}{d\hat{t}^2} + k\hat{x}x_0 = 0 \\ &\implies \frac{m}{(m/k)}\frac{d^2\hat{x}}{d\hat{t}^2} + k\hat{x} = 0 \\ &\implies k\frac{d^2\hat{x}}{d\hat{t}^2} + k\hat{x} = 0 \\ &\implies \frac{d^2\hat{x}}{d\hat{t}^2} + \hat{x} = 0 \end{aligned}$$

and we get our "universal problem". This is a second order ODE and it is easier to solve than the standard oscillator equation since its characteristic polynomial is:

$$r^2 + 1 = 0$$

which clearly has roots $\pm i$. This gives the general solution:

$$\hat{x}(\hat{t}) = c_1\cos(\hat{t}) + c_2\sin(\hat{t})$$
$$\hat{x}'(\hat{t}) = -c_1\sin(\hat{t}) + c_2\cos(\hat{t})$$

Using our velocity condition gives:

$$\hat{x}'(0) = (0)/x_0 = 0 = c_2 \implies \hat{x}(\hat{t}) = c_1\cos(\hat{t})$$

And our IC gives:

$$\hat{x}(0) = x_0/x_0 = c_1$$

Giving the particular solution:

$$\hat{x}(\hat{t}) = \cos(\hat{t})$$

Now lets change back to dimensional variables:

$$\begin{aligned} \hat{x}(\hat{t}) = \cos(\hat{t}) &\implies \frac{x}{x_0} = \cos(\frac{t}{\tau}) \\ &\implies x(t) = x_0\cos(\frac{t}{\tau}) \end{aligned}$$

Which recovers our dimensional solution.

Example 2: Let's now add some damping again (see equation 1.5)

- Pick a system of units for x and t. What is the resulting dimensionless equation?
- Show that this time there is 1 nondimensional parameter that appears. What is its physical interpretation?
- What happens mathematically when this parameter is very small or very big? What does it correspond to physically?

Solution: Starting from the equation (1.6):

$$m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = 0 \tag{1.5}$$

Choose characteristic units:

$$T = \sqrt{\frac{m}{k}}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{x} = \frac{x}{X}$$

Derivatives transform as:

$$\frac{dx}{dt} = \frac{X}{T}\frac{d\tilde{x}}{d\tilde{t}}, \quad \frac{d^2x}{dt^2} = \frac{X}{T^2}\frac{d^2\tilde{x}}{d\tilde{t}^2}$$

Substitute into equation (1.5):

$$m\left(\frac{X}{T^2}\frac{d^2\tilde{x}}{d\tilde{t}^2}\right) + \lambda\left(\frac{X}{T}\frac{d\tilde{x}}{d\tilde{t}}\right) + kX\tilde{x} = 0$$

Simplify using $T = \sqrt{\frac{m}{k}}$:

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} + \beta\frac{d\tilde{x}}{d\tilde{t}} + \tilde{x} = 0$$

where

$$\beta = \frac{\lambda}{\sqrt{km}}$$

The single nondimensional parameter β represents the relative damping of the system. It indicates how significant the damping λ is compared to the system's inertia and stiffness.

- **Small** β **(**$\beta \ll 1$**):** Damping is negligible. - *Mathematically:* The equation approximates to simple harmonic motion. - *Physically:* The system is underdamped and oscillates freely.

- **Large** β **(**$\beta \gg 1$**):** Damping dominates. - *Mathematically:* The system's response slows down exponentially without oscillations. - *Physically:* The system is overdamped, returning to equilibrium slowly.

Example 3: Consider the advection-diffusion equation (1.4) on the interval $[0, L]$.

- Show that there are two characteristic timescales associated with the input parameters.
- Non-dimensionalize the problem using L as a unit length, and the timescale associated with v as the unit time.
- How many non-dimensional parameters appear in the problem? What is the physical interpretation of the parameter(s)?
- What happens mathematically when the parameter(s) is/are very small or very big? What does it correspond to physically?

Solution: We consider the input parameters v and D, where $[v] = \frac{L}{T}$ and $[D] = \frac{L^2}{T}$. From here we can derive the characteristic timescales:

$$v = \frac{L}{T_1} \implies T_1 = \frac{L}{v}$$
$$D = \frac{L^2}{T_2} \implies T_2 = \frac{L^2}{D}$$

Now considering timescale T_1 associated with v, we nondimensionalize the problem such that:

$$\hat{x} = \frac{x}{L} \implies x = \hat{x}L$$
$$\hat{t} = \frac{tv}{L} \implies t = \frac{\hat{t}L}{v}$$

Now through substitution we obtain:

$$\frac{\partial T}{\partial(\frac{\hat{t}L}{v})} + v\frac{\partial T}{\partial(\hat{x}L)} = D\frac{\partial^2 T}{\partial(\hat{x}L)^2}$$

$$\frac{v}{L}\frac{\partial T}{\partial \hat{t}} + \frac{v}{L}\frac{\partial T}{\partial \hat{x}} = \frac{D}{L^2}\frac{\partial^2 T}{\partial \hat{x}^2}$$

$$\frac{\partial T}{\partial \hat{t}} + \frac{\partial T}{\partial \hat{x}} = \frac{D}{Lv}\frac{\partial^2 T}{\partial \hat{x}^2}$$

Thus we've obtained one dimensionless parameter denoting the rate of diffusion. When the parameter is large, we see that heat diffuses faster, whereas when the parameter is small, heat diffuses slower.

1.4 The Buckingham π theorem

As we have seen through examples, by non-dimensionalizing the problem, we can systematically reduce the number of parameters in the equations and/or boundary conditions. This is a very general result, which has been formalized in the *Buckingham* π theorem.

Loosely speaking, the theorem states that the number of *independent* dimensionless parameters of a problem is equal to the number of independent (relevant) dimensional parameters, minus the number of fundamental dimensions in the problem.

Example 1: In the harmonic oscillator problem, we have 3 relevant independent parameters: x_0, m, and k. There are 3 fundamental dimensions involved: mass (from m), length (from x_0) and time (from k). Therefore, the system can be non-dimensionalized so that there are $3 - 3 = 0$ remaining dimensionless parameters.

Example 2: Consider now the damped harmonic oscillator and its initial conditions.

- Count the number of independent parameters, and fundamental dimensions of the problem.
- What is the minimum number of dimensionless parameters needed to represent it?
- Compare this with your findings of the previous section.

Solution: Considering the damped harmonic oscillator, there are 4 independent parameters (m, t_0, k, λ) and 3 fundamental dimensions (mass,time,length). Thus, there will be $4 - 3 = 1$ independent dimensionless parameters.

Example 3: Consider the advection-diffusion equation.

- Count the number of independent parameters, and fundamental dimensions of the problem.
- What is the minimum number of dimensionless parameters needed to represent it?
- Compare this with your findings of the previous section.

Solution: We consider the advection-diffusion equation:

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = D\frac{\partial^2 T}{\partial x^2}$$

on the domain $[0, L]$ with $T(0) = T_0$ We have independent parameters with dimensions:

- v with dimensions length/time
- D with dimensions length^2/time
- T_0 with dimension of temperature

- L with dimension of length

We have 4 independent parameters, and 3 fundamental dimensions, thus by the Buckingham π theorem we have $4 - 3 = 1$ independent dimensionless parameters. This follows the results from section 1.3 example 3, where we found one dimensionless parameter denoting the rate of diffusion.

The Buckingham π theorem is very important, because it shows that non-dimensionalization can always be used to reduce the number of input parameters of the problem – which is very convenient if you want to avoid wasting time exploring a parameter space that is bigger than necessary. It also shows that two problems that are apparently very different, but have the *same* dimensionless form, will behave in exactly the same way. We can therefore build 'universal' non-dimensional solutions that depend only on the non-dimensional parameters, and then recover the desired dimensional solutions (if needed) for a wide range of dimensional input parameters.

Example: In fluid mechanics, it can be shown that the incompressible (non-rotating, unstratified, non-magnetic) flow past an object satisfies a universal equation that depends only on 1 dimensionless parameter: the Reynolds number, $\text{Re} = UL/\nu$ where U is the characteristic velocity of the incoming flow, L a characteristic size of the object, and ν the viscosity of the fluid. Therefore, in order to test the aerodynamic properties of an airplane or a race car in air at velocity U, we simply have to create a small-scale model of that airplane or car of size L/a (where $a > 1$), and put it in a wind-tunnel with a wind of velocity aU. This works, because their Reynolds numbers are the same.

1.5 What is large, and what is small?

Non-dimensionalizing a problem is also very useful, because it helps us have a formal way of defining concepts such as 'large' and 'small' quantities.

Indeed, a quantity is only 'large' or 'small' relative to the system of units used: 1g, is also 0.001kg (small compared to a kg), but is also 1000μg (large compared to a μg) – so is 1g large, or is it small? That depends on your system of units!

This then leads to the much more philosophical question of what is the *correct* system of units to use for a given problem. As the example above shows, using real physical units (like cgs or SI) can confuse the matter, because a quantity can be large *or* small depending on the choice made.

However, we learned in the previous sections that there are intrinsic length, time, mass, velocity, etc. scales that can be created from the system parameters, and these are often a much more meaningful choice for the system of units. In these new units, we can then determine more objectively whether a quantity is large or small.

Example 1: In the example of the damped harmonic oscillator, how would you quantify if the effect of damping is large or small?

Solution: To quantify whether the effect of damping is large or small, we first nondimensionalize the equation using

$$\hat{x} = \frac{x}{x_0} \implies x = x_0\hat{x}$$
$$\hat{t} = \frac{t}{T_c} = \frac{t}{\sqrt{\frac{m}{k}}} \implies t = \hat{t}m^{1/2}k^{-1/2}$$

Substituting these into the equation, we obtain:

$$m\frac{d^2(x_0\hat{x})}{d(\hat{t}m^{1/2}k^{-1/2})^2} = -k(x_0\hat{x}) - \lambda\frac{d(x_0\hat{x})}{d(\hat{t}m^{1/2}k^{-1/2})}$$
$$\frac{mx_0k}{m}\frac{d^2\hat{x}}{d\hat{t}^2} = -kx_0\hat{x} - x_0\frac{\lambda k^{1/2}}{m^{1/2}}\frac{d\hat{x}}{d\hat{t}}$$
$$\frac{d^2\hat{x}}{d\hat{t}^2} = -x - \frac{\lambda}{\sqrt{mk}}\frac{d\hat{x}}{d\hat{t}}$$

We consider the damping coefficient:

$$\varphi = \frac{\lambda}{\sqrt{mk}}$$

The effect of damping can be quantified by the size of φ: if $\varphi \gg 1$ damping is rapid, but if $\varphi \ll 1$ damping is slow.

Example 2: In the example of the time-dependent advection-diffusion equation, how would you quantify if the effect of diffusion is large or small?

Solution:

To quantify if the effect of diffusion is large or small in the time-dependent advection-diffusion equation, we compare the characteristic timescales of advection and diffusion:

- **Advection timescale:** $T_{\text{adv}} = \dfrac{L}{v}$
- **Diffusion timescale:** $T_{\text{diff}} = \dfrac{L^2}{D}$

We define the dimensionless **Péclet number** (Pe) as the ratio of these timescales:

$$\text{Pe} = \frac{T_{\text{diff}}}{T_{\text{adv}}} = \frac{Lv}{D}$$

This parameter quantifies the relative importance of diffusion and advection:

- If $\text{Pe} \gg 1$, advection dominates (D is small), so diffusion effects are negligible.
- If $\text{Pe} \ll 1$, diffusion dominates (D is large), so diffusion effects are significant.

Therefore, by evaluating the Péclet number, we can determine whether the effect of diffusion is large or small in the system.

1.6 Take-home messages

Here are a few things to remember from this lecture:

- Dimensional analysis can help you discover important characteristic scales of a problem.
- These scales can be used to form a new system of units for your equations and boundary/initial conditions.
- Non-dimensionalizing equations and boundary/initial conditions using these scales reduces the dimensionality of parameter space (Buckingham π theorem).
- It also helps you find out objectively if a quantity is large or small compared with these intrinsic problem scales. This will be particularly useful when we start doing some asymptotic analysis, which is the study of equations that have very large or very small parameters.

Chapter 2

Partial Differential Equations

Lecture 2 is edited by Victoria, Janice, Julian and Howard.

We now begin one of the three main sections of this course, on partial differential equations (PDEs). The first part of this chapter will mostly review what you will have learned in an undergraduate PDE class, after which we will move on to more advanced concepts.

2.1 Definitions

In what follows, we will work with differential operators (say, $\mathcal{D}$) on the space of functions. An *ordinary* differential equation (ODE) is of the form $\mathcal{D}f = 0$, where f is a function of a *single* variable, and $\mathcal{D}$ therefore only involves regular derivatives with respect to that variable. A *partial* differential equation (PDE) is also of the form $\mathcal{D}f = 0$, but this time f is a function of *multiple* variables, and $\mathcal{D}$ therefore involves partial derivatives with respect to these variables.

Linear operators: An operator is said to be linear (in which case it is often named $\mathcal{L}$) if, for any two functions f and g and scalar a, we have

$$\mathcal{L}(f+g) = \mathcal{L}f + \mathcal{L}g \tag{2.1}$$

$$\text{and } \mathcal{L}(af) = a\mathcal{L}f \tag{2.2}$$

The second condition ensures that the operator $\mathcal{L}$ is also homogeneous (see below).

Example of linear operators:

- For ODEs: the integral $\int (f+g)(x)dx = \int f(x)dx + \int g(x)dx$ and $\int af(x)dx = a\int f(x)dx$
- For PDEs: the laplacian $\Delta(f+g)(\mathbf{x}) = \Delta f(\mathbf{x}) + \Delta g(\mathbf{x})$ and $\Delta(af)(\mathbf{x}) = a\Delta f(\mathbf{x})$

We see that $\mathcal{L}f$ is always a linear combination of f and its derivatives (regular or partial).

Homogeneneous vs. non-homogeneous linear equations: Linear ODEs or PDEs of the form $\mathcal{L}f = 0$, where $\mathcal{L}$ is a linear operator, are automatically homogeneous. We see that they are trivially satisfied by the null function $f \equiv 0$. Linear ODEs and PDEs of the form $\mathcal{L}f = F$ where F is some explicit function of *only* the independent variables are not homogeneous. Notably, $f = 0$ is not a solution of these equations.

Examples of homogeneous and non-homogeneous linear equations:

- homogeneous: $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ (Laplace's eqn.)
- nonhomogeneous: $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = F(x,y)$ (Poisson's eqn.)

Nonlinear operators: A nonlinear operator is simply not linear. Most ODEs and PDEs in the 'real' world involve nonlinear operators. However, analytical solutions of nonlinear ODEs and PDEs are rare, so this lecture will mostly focus on linear PDEs. If you are dealing with a nonlinear PDE, you will likely have to solve it numerically (if you need exact solutions) or approximately (using, e.g. local linearization or tools from asymptotic analysis, see Chapter 3).

Examples of famous nonlinear PDEs:

- Navier-Stokes: $\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u}$
- Burgers' equation: $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \nu \dfrac{\partial^2 u}{\partial x^2}$

Initial / boundary conditions: In addition to the differential equation itself, a real-world application will usually also involve initial conditions or boundary conditions that must be applied to find the solution of the problem.

- *Initial* conditions usually refer to conditions applied at some given point in time everywhere in space.
- *Boundary* conditions usually refer to conditions applied on the spatial domain boundaries at all times.

Just like we did for the ODE or PDE itself, we will distinguish between linear and nonlinear boundary conditions.

Homogeneous boundary conditions: Homogeneous linear boundary conditions are boundary conditions that are trivially satisfied by the null function. They can be expressed as a linear combination of the function and its derivative(s) being zero on the boundary.

Examples:

- Homogeneous Dirichlet conditions: $f(\mathbf{x}) = 0, \forall \mathbf{x} \in \partial\Omega$
- Homogeneous von Neumann conditions: $\hat{\mathbf{n}}(\mathbf{x}) \cdot \nabla f(\mathbf{x}) = 0, \forall \mathbf{x} \in \partial\Omega, \quad \hat{\mathbf{n}}$ normal to $\partial\Omega$
- Homogeneous Robin conditions: $\alpha f + \beta \hat{\mathbf{n}} \cdot \nabla f = 0, \forall \mathbf{x} \in \partial\Omega, \quad \hat{\mathbf{n}}$ normal to $\partial\Omega$

Dimension of a PDE: Confusingly, we also use the terminology 'dimension' (see previous lecture) to denote the number of independent variables of a PDE. The meaning of 'dimension' should hopefully be clear from the context in which it is used.

Examples:

- Example of a 2D PDE: $u_t = u_{xx}$, with $u(x,t)$ (1D heat equation)
- Example of a 3D PDE: $u_{tt}(\mathbf{x},t) = c^2 \Delta u(\mathbf{x},t)$, where $\mathbf{x} = (x,y)$ (2D wave equation)
- The Boltzmann equation: $\dfrac{\partial f}{\partial t} + \dfrac{\mathbf{p}}{m} \cdot \nabla f + \mathbf{F} \cdot \dfrac{\partial f}{\partial \mathbf{p}} = \left(\dfrac{\partial f}{\partial t}\right)_{\text{coll}}$,
  where $f(\mathbf{x},\mathbf{p},t)$ with position $\mathbf{x} = (x,y,z)$, momentum $\mathbf{p} = (p_x,p_y,p_z)$, and time t. (7 dims)

Order of an ODE or PDE: The order of a linear differential equation is equal to the highest order derivative appearing in the operator $\mathcal{L}$.

Examples:

- heat equation: $u_t(\mathbf{x},t) = \Delta u(\mathbf{x},t)$ (2^{nd} order)
- wave equation: $u_{tt}(\mathbf{x},t) = \Delta u(\mathbf{x},t)$ (2^{nd} order)
- $\dfrac{\partial f}{\partial t} = c\dfrac{\partial^4 f}{\partial x^4}$ (4^{th} order)

With all these definitions now established, we dive in this Chapter into an important class of PDEs, namely second order, 2D, linear PDEs, which have been studied extensively as they very commonly arise in physics and engineering. We will not touch first order PDEs, which you should have seen in an undergraduate-level PDE course. We will cover 3D and 4D linear PDEs later in this course.

2.2 Second order 2D linear PDEs (the basics)

2.2.1 Classification of PDEs

A second order, 2D linear PDE in two variables (x, y) can be written, in all generality, as

$$a(x,y)\frac{\partial^2 f}{\partial x^2} + 2b(x,y)\frac{\partial^2 f}{\partial x \partial y} + c(x,y)\frac{\partial^2 f}{\partial y^2}$$
$$+d(x,y)\frac{\partial f}{\partial x} + e(x,y)\frac{\partial f}{\partial y} + g(x,y)f + h(x,y) = 0 \quad (2.3)$$

If the PDE is homogeneous, we further have $h(x, y) \equiv 0$. The first line of this equation, which contains the highest-order derivatives, is called the *principal part*.

Using the theory of canonical forms, it is possible to show that second order 2D linear PDEs can be classified into 3 canonical types: parabolic equations, hyperbolic equations and elliptic equations, each of which has distinct properties. Furthermore, that classification only depends on the PDE's principal part. More specifically, compute the local discriminant of the PDE as

$$\Delta(x,y) = b^2(x,y) - a(x,y)c(x,y) \quad (2.4)$$

- If $\Delta(x, y) < 0$ at (x, y) the PDE is locally elliptic
- If $\Delta(x, y) > 0$ at (x, y) the PDE is locally hyperbolic
- If $\Delta(x, y) = 0$ at (x, y) the PDE is locally parabolic

Note how some equations can have a hyperbolic nature in one part of a domain, and an elliptic nature in the other. But parabolic equations are usually parabolic everywhere.

Example: Consider the standard PDES with constant coefficients:

- the diffusion equation
$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}, \quad (2.5)$$
- the wave equation
$$\frac{\partial^2 f}{\partial t^2} = c_w^2\frac{\partial^2 f}{\partial x^2}, \quad (2.6)$$
- Laplace's equation
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad (2.7)$$

What are their types?

Solution:
The diffusion equation:

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \Rightarrow D\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0$$

This gives $a = D$, $b = 0$, and $c = 0$. Therefore,

$$\Delta = b^2 - ac = 0 - D(0) = 0$$

This result shows that the diffusion equation is parabolic.

The wave equation:

$$\frac{\partial^2 f}{\partial t^2} = c_w^2\frac{\partial^2 f}{\partial x^2} \Rightarrow c + w^2\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} = 0$$

This gives $a = c_w^2$, $b = 0$, and $c = -1$. Therefore,

$$\Delta = b^2 - ac = 0 - (c_w^2)(-1) = c_w^2 > 0$$

This result shows that the wave equation is hyperbolic.

Laplace's equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

This gives $a = 1$, $b = 0$, and $c = 1$. Therefore,

$$\Delta = b^2 - ac = 0 - (1)(1) = -1 < 0$$

This result shows that Laplace's equation is elliptic.

In fact, the theory of canonical forms states that it is always possible to find a change of variables that transforms a linear second order 2D PDE into one whose *principal part* has the same form as the diffusion equation (if it is parabolic), the wave equation (if it is hyperbolic) or Laplace's equation (if it is elliptic). That is why studying these three equations is so fundamental to the theory of second order 2D linear PDEs.

In this what follows, we now focus on homogeneous linear PDEs in a domain that is bounded in at least one of the spatial variables (i.e. at least one of the spatial variables lives on a finite interval). The boundary conditions on that interval are assumed to be homogeneous. A powerful technique for solving (some) equations of this type is called the *method of separation of variables*. We first discuss the method in general, and then solve a few simple PDEs to see how it works in practice.

2.2.2 Method of separation of variables (general idea)

The method of separation of variables is only appropriate for certain types of 'separable' linear PDEs with appropriately 'separable' boundary conditions. In order to be separable, a 2D homogeneous PDE (i.e. a PDE in 2 independent variables, let's call them x and y) must be such that it is possible to rewrite it as

$$\mathcal{L}_x f = \mathcal{L}_y f, \tag{2.8}$$

where $\mathcal{L}_x$ is a linear operator that only includes partial derivatives in the x variable, and $\mathcal{L}_y$ only includes partial derivatives in the y variable.

Examples of separable and non-separable linear second order 2D PDEs:

- **Separable:**
  - Laplace's Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
  - Poisson's Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$
  - Diffusion Equation: $\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$
- **Non-Separable:**
  - Advection-Diffusion Equation: $\frac{\partial u}{\partial t} + v_x(x, y)\frac{\partial u}{\partial x} + v_y(x, y)\frac{\partial u}{\partial y} = D \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$

More generally, if $b(x, y) = h(x, y) = 0$ in (2.3), and all of the other 'coefficients' $a(x, y)$, ... $g(x, y)$ of the PDE are constant, then it is homogeneous and separable.

If the boundary conditions are homogeneous as assumed, in order to be separable as well the domain boundaries must simply be composed of lines or curves where a given independent variable is held constant.

Examples of separable and non-separable homogeneous boundary conditions:

- **Separable:**

- Laplace's equation in Cartesian coordinates on a rectangular plate with $x \in (0, L)$ and $y \in (0, H)$, with homogeneous Dirichlet conditions on all four sides.
- Laplace's equation in polar coordinates (see later) on a disk $D = \{(r, \theta) \in \mathbb{R} | 0 \leq r \leq R, 0 \leq \theta \leq 2\pi\}$ with $f(R, \theta) = 0$.

- **Non-Separable:**
  - Laplace's equation in Cartesian coordinates on a parallelogram with homogeneous boundary conditions on all four sides.
  - Laplace's equation in Cartesian coordinates on a disk $D = \{(x, y) \in \mathbb{R} | x^2 + y^2 = R^2\}$ with $f(x, y) = 0 \ \forall (x, y) \in \delta\Omega$ (along the boundary).

We therefore note that not all linear homogeneous PDEs are separable, and even if the PDE itself is, not all homogeneous boundary conditions are separable either (this depends on the domain shape with respect to the coordinate system selected).

If the problem is separable, then we can often solve it by leveraging the homogeneity and linearity of both equation and boundary conditions. Indeed, if (2.8) is satisfied, then there are (probably) separable solutions to the problem of the form $f(x, y) = A(x)B(y)$, satisfying

$$B(y)\mathcal{L}_x A = A(x)\mathcal{L}_y B \tag{2.9}$$

Dividing by AB, we obtain

$$\frac{\mathcal{L}_x A}{A(x)} = \frac{\mathcal{L}_y B}{B(y)} \tag{2.10}$$

Written in this way, we now see that the left-hand side only depends on x, while the right-hand side only depends on y, and that can only be possible if both are exactly constant:

$$\frac{\mathcal{L}_x A}{A(x)} = \lambda = \frac{\mathcal{L}_y B}{B(y)} \tag{2.11}$$

For this to be a solution, we then need at the same time

$$\mathcal{L}_x A = \lambda A \text{ and } \mathcal{L}_y B = \lambda B. \tag{2.12}$$

In other words, A must be an eigenfunction of $\mathcal{L}_x$, and B must be an eigenfunction of $\mathcal{L}_y$, and they must share the eigenvalue λ in order for $f = AB$ to be a solution of the PDE. That eigenvalue usually depends on the boundary conditions applied, and often, many such triplets (A,B,λ) actually exist, so the true solution of the PDE would involve a linear combination of these individual separable solutions, and further work is needed to ensure that all the boundary and/or initial conditions are satisfied. Whether such a solution to the full problem always exist or not depends on the nature of the PDE, and is the focus of Sturm-Liouville theory (see later in the course).

In the next section, we look at a few examples of how to apply this method in practice.

2.2.3 The diffusion equation in a finite interval

Let's consider the diffusion equation

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \tag{2.13}$$

on the interval $(0, L)$, with homogeneous von Neumann conditions. At time $t = 0$, $f(x, 0) = f_0(x)$, where $f_0(x)$ satisfies the boundary conditions as well.

- Show, using separation of variables, that the general solution of the equation can be written in the form

$$f(x, t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-t/\tau_n} \tag{2.14}$$

  where you need to determine what τ_n is.

- Use the orthogonality properties of the cosine functions to apply the initial conditions and find the coefficients c_n.

- What is the limit of $f(x,t)$ as $t \to \infty$? What is the mathematical meaning of this value? What is the physical meaning of this value?

Note: the orthogonality condition for cosines is :

$$\begin{aligned}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx \quad &= 0 \text{ if } m \neq n \\ &= \frac{L}{2} \text{ if } m = n > 0, \\ &= L \text{ if } m = n = 0 \end{aligned} \tag{2.15}$$

Solution:

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \tag{2.16}$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \text{ when } x = 0, L \\ f(x,0) = f_0(x) \end{cases} \tag{2.17}$$

Assume $f(x,t) = A(x)B(t)$. Then,

$$\frac{\partial f}{\partial t} = A\frac{dB}{dt} \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = B\frac{d^2A}{dx^2} \tag{2.18}$$

$$\Rightarrow \frac{1}{B}\frac{dB}{dt} = \frac{D}{A}\frac{d^2A}{dx^2} = \lambda \tag{2.19}$$

$$\Rightarrow \begin{cases} \frac{d^2A}{dx^2} = \frac{\lambda}{D}A \\ \frac{dB}{dt} = \lambda B \end{cases} \tag{2.20}$$

Consider the sign of λ. Note that this is a diffusion problem, so we expect that the solution will trend towards some steady-state and won't grow exponentially with respect to time. So, given $\frac{dB}{dt} = \lambda B$ with solution $B(t) = be^{\lambda t}$, we expect $\lambda \leq 0$.

Case $\lambda = 0$: Consider $\frac{d^2A}{dx^2} = \frac{\lambda}{D}A$. Then,

$$\Rightarrow \frac{d^2A}{dx^2} = 0 \tag{2.21}$$

Applying boundary condition $\frac{dA}{dx} = 0$ at $x = 0, L$ gives us the solution $A_0(x) = constant$. Similarly, $B_0(t) = constant$.

Case $\lambda < 0$: Consider $\frac{d^2A}{dx^2} = \frac{\lambda}{D}A$. Then,

$$A(x) = \begin{cases} \sin\left(\sqrt{\frac{-\lambda}{D}}x\right) \\ \cos\left(\sqrt{\frac{-\lambda}{D}}x\right) \end{cases} \tag{2.22}$$

Applying boundary condition $\frac{dA}{dx} = 0$ at $x = 0, L$, we have:

$$A(x) = \cos\left(\sqrt{\frac{-\lambda}{D}}x\right) \quad \text{to satisfy} \quad \left.\frac{dA}{dx}\right|_{x=0} = 0 \tag{2.23}$$

and:

$$\frac{dA}{dx} = -\sqrt{\frac{-\lambda}{D}}\sin\left(\sqrt{\frac{-\lambda}{D}}x\right) \tag{2.24}$$

$$\Rightarrow \frac{dA}{dx}\bigg|_{x=L} = -\sqrt{\frac{-\lambda}{D}} \sin\left(\sqrt{\frac{-\lambda}{D}} L\right) = 0 \tag{2.25}$$

$$\Rightarrow \sqrt{\frac{-\lambda}{D}} L = n\pi \tag{2.26}$$

$$\lambda_n = -\frac{n^2\pi^2 D}{L^2} \tag{2.27}$$

Summarizing up to this point, we have:

$$\begin{cases} A_n(x) = a_n \cos\left(\frac{n\pi x}{L}\right) \\ B_n(t) = b_n e^{-t/\tau_n} \end{cases} \tag{2.28}$$

with $\tau_n = \frac{-1}{\lambda_n} = \frac{L^2}{n^2\pi^2 D}$

Combining our $A(x)$ and $B(t)$ terms, we have:

$$f(x,t) = a_0 b_0 + \sum_{n=1}^{\infty} a_n b_n \cos\left(\frac{n\pi x}{L}\right) e^{-t/\tau_n} \tag{2.29}$$

Simplifying constants:

$$f(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-t/\tau_n} \tag{2.30}$$

Note that in this summation we are leveraging the fact that the boundary conditions are homogeneous.

Now, we apply the initial conditions to solve for the coefficients. We want $f(x,0) = f_0(x)$:

$$f(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) = f_0(x) \tag{2.31}$$

We project this expression onto the cosines:

$$\int_0^L \left[c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right)\right] \cos\left(\frac{m\pi x}{L}\right) dx \tag{2.32}$$

Using the orthogonality condition for cosine to each term in the summation, we have::

$$\int_0^L c_0 \cos\left(\frac{m\pi x}{L}\right) dx + \sum_{n=1}^{\infty}\left[c_n \int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx\right] \tag{2.33}$$

$$\Rightarrow 0 + \frac{L}{2} c_m = \int_0^L f_0(x) \cos\left(\frac{m\pi x}{L}\right) dx \tag{2.34}$$

$$\Rightarrow c_m = \frac{2}{L}\int_0^L f_0(x) \cos\left(\frac{m\pi x}{L}\right) dx \tag{2.35}$$

when $m \neq 0$, and

$$0 + Lc_0 = \int_0^L f_0(x) dx \tag{2.36}$$

$$\Rightarrow c_0 = \frac{1}{L}\int_0^L f_0(x) dx \tag{2.37}$$

when $m = 0$.

All together, we have the solution

$$f(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-t/\tau_n} \tag{2.38}$$

with

$$\tau_n = \frac{-1}{\lambda} = \frac{L^2}{n^2\pi^2 D} \tag{2.39}$$

and coefficients

$$\begin{cases} c_0 = \frac{1}{L}\int_0^L f_0(x)dx \\ c_n = \frac{2}{L}\int_0^L f_0(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad \text{when } n \neq 0 \end{cases} \tag{2.40}$$

Consider how the solution diffuses. We have that

$$\lim_{t\to\infty} f(x,t) = c_0 = \frac{1}{L}\int_0^L f_0(x)dx \tag{2.41}$$

which is the mean of $f_0(x)$ over the domain $(0, L)$. Because of the no flux conditions (von Neumann conditions), the total amount of f in the domain is conserved.

Note how:

- The function $f(x,t)$ is decomposed into a sum of Fourier 'modes' that each has its own spatial dependence (the eigenmode $A_n(x)$) and its own decay timescale τ_n.
- Higher-order modes (higher n) capture finer spatial scales, and these decay faster. After a large amount of time, only the largest system scales remain.
- Both these behaviors are fundamental behaviors of the diffusion equation.
- Adding all of the individual separable solutions together was only possible because the boundary conditions are homogeneous!

2.2.4 Take-home messages for lecture 2

Here are a few things to remember:

- Be familiar with the definitions (linear; homogeneous; dimensions; order; types of boundary conditions; the three canonical 2nd order 2D PDEs)
- The method of separation of variables is powerful, but requires that (1) the PDE and boundary conditions be linear, (2) the PDE and boundary conditions be separable, (3) that the boundary conditions be homogeneous for at least one of the variables.

2.2.5 The wave equation in a finite interval

Lecture edited by Charlie, Dante and Yiqin.

Let's consider the wave equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} \tag{2.42}$$

on the interval $(0, L)$, with homogeneous Dirichlet conditions. At time $t = 0$, $f(x, 0) = f_0(x)$, where $f_0(x)$ satisfies the boundary conditions as well, and $\partial f/\partial t(x, 0) = 0$.

- Show, using separation of variables, that the general solution of the equation can be written in the form

$$f(x,t) = \sum_{n=1}^{\infty} [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)] \sin\left(\frac{n\pi x}{L}\right) \tag{2.43}$$

  where you need to determine what ω_n is.

- Use the orthogonality properties of the sin functions to apply the initial conditions and find the coefficients a_n and b_n.

Note: the orthogonality condition for sines is :

$$\begin{aligned} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \quad &= 0 \text{ if } m \neq n \\ &= \frac{L}{2} \text{ if } m = n \end{aligned} \tag{2.44}$$

Solution: Assume

$$f(x,t) = A(x)B(t)$$

send it to original equation, get

$$\frac{d^2B(t)}{dt^2}A(x) = c^2 \frac{d^2A(x)}{dx^2}B(t)$$

$$\frac{1}{c^2B(t)}\frac{d^2B(t)}{dt^2} = \frac{1}{A(x)}\frac{d^2A(x)}{dx^2}$$

so both sides of the equation must be equal to a constant λ:

$$\frac{d^2A(x)}{dx^2} = \lambda A(x)$$

$$\frac{d^2B(t)}{dt^2} = \lambda c^2 B(t)$$

if $\lambda = 0$,

then $A_0(x)$ and $B_0(t)$ should in the form $A_0(x) = ax + a_0$ and $B_0(t) = bt + b_0$, based on the boundary conditions, $f(0,t) = f(L,t) = 0$ thus $a = a_0 = b = b_0 = 0$

if $\lambda < 0$, the solution to the spatial problem is of the form

$$A(x) = a\sin(\sqrt{-\lambda}x) + b\cos(\sqrt{-\lambda}x)$$

Then consider the homogeneous Dirichlet conditions $A(0) = 0$ means that $b = 0$, so we can set $A(x) = a\sin(\sqrt{-\lambda}x)$. The function A also satisfies

$$A(L) = a\sin(\sqrt{-\lambda}L) = 0$$

so

$$\sqrt{-\lambda}L = n\pi \rightarrow \lambda_n = -\frac{n^2\pi^2}{L^2}$$

Putting this together we have

$$A_n(x) = a_n \sin(\frac{n\pi x}{L})$$

. Finally, to find the solution for $B(t)$ we plug λ into B equation,

$$\frac{d^2B(t)}{dt^2} = \lambda c^2 B(t) = -\frac{n^2\pi^2}{L^2} c^2 B(t)$$

which has solutions

$$B_n(t) = a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L})$$

Each separable solution is therefore of the form:

$$f_n(x,t) = A_n(x)B_n(t) = [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)] \sin(\frac{n\pi x}{L})$$

with

$$\omega_n = n\pi c/L$$

Finally the full solution is a linear combination of all of these separable soltuions:

$$f(x,t) = \sum_{n=1}^{\infty} [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)] \sin\left(\frac{n\pi x}{L}\right)$$

To find the coefficients a_n and b_n we apply the initial conditions:

$$f(x,0) = f_0(x) = \sum_{n=1}^{\infty} a_n sin(\frac{n\pi x}{L}) \qquad (*)$$

$$\frac{\partial f}{\partial t(x,0)} = 0 = \sum_{n=1}^{\infty} b_n \omega_n \sin(\frac{n\pi x}{L})$$

. The second equation reveals $b_n = 0$. To get the a_n we apply orthogonality condition to $(*)$:

$$\int_0^L (*) \sin(\frac{m\pi x}{L}) dx$$

$$\int_0^L f_0(x) \sin(\frac{m\pi x}{L}) dx = \frac{La_m}{2}$$

so

$$a_m = \frac{2}{L} \int_0^L f_0(x) \sin(\frac{m\pi x}{L}) dx$$

Note how:

- The function $f(x,t)$ is again decomposed into a sum of Fourier 'modes' that each has its own spatial dependence and this time its own *oscillation* frequency ω_n.
- Higher-order modes (higher n) capture finer spatial scales, and these oscillate faster.
- Both these behaviors are fundamental behaviors of the wave equation.
- Note how here $\omega_n = n\omega_0$ where ω_0 is the fundamental oscillation frequency. The fact that the frequencies are integer multiples of one another is the reason music exists!

2.2.6 Laplace's equation in a finite domain

Consider Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{2.45}$$

on a rectangular plate with $x \in (0, L)$ and $y \in (0, H)$, with homogeneous Dirichlet conditions on the three sides $y = 0$, $y = H$, and $x = 0$ and boundary conditions $f(L, y) = f_0(y)$ on the fourth side ($x = L$).

- Show, using separation of variables, that the general solution of the equation can be written in the form

$$f(x,t) = \sum_{n=1}^{\infty} \left[a_n \cosh\left(\frac{x}{d_n}\right) + b_n \sinh\left(\frac{x}{d_n}\right)\right] \sin\left(\frac{y}{d_n}\right) \tag{2.46}$$

where you need to determine what d_n is.

- Explain why the Fourier Series is in the y rather than the x variable.

- Apply the boundary condition at $x = 0$ and $x = L$ to find a_n and b_n

Solution:

We start by assuming

$$f(x,y) = A(x)B(y)$$

With this, the boundary conditions

$$f(x,0) = f(x,H) = f(0,y) = 0$$

become

$$A(0) = B(0) = B(H) = 0$$

Substituting this into the Laplace equation we have

$$\frac{1}{A(x)}\frac{d^2A(x)}{dx^2} = -\frac{1}{B(y)}\frac{d^2B(y)}{dy^2}$$

so both sides of the equation must be equal to a constant λ:

$$\frac{d^2A(x)}{dx^2} = \lambda A(x)$$

$$\frac{d^2B(y)}{dy^2} = -\lambda B(y)$$

.

$$A(x) = a\cosh(\sqrt{\lambda}x) + b\sinh(\sqrt{\lambda}x)$$

$$B(y) = \sin(\sqrt{-\lambda}y)$$

, where we have used the fact that $B(0) = 0$ to eliminate the cosine solution as in the previous example. Then, the other boundary condition in y

$$B(H) = \sin(\lambda H) = 0$$

implies

$$\lambda H = n\pi$$

so

$$\lambda = \frac{n\pi}{H}$$

The boundary condition at $x = 0$ implies

$$A(0) = a = 0$$

, so only the sinh solution is left. The solution is a linear combination of all of the individual separable solutions found:

$$f(x,y) = \sum_{n=1}^{\infty} b_n \sinh(\frac{x}{d_n})\sin(\frac{y}{d_n})$$

with

$$d_n = 1/\lambda = \frac{H}{n\pi}$$

Finally we apply the last boundary condition:

$$f(L,y) = f_0(y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi L/H)\sin(n\pi y/H)$$

Projecting on the sine modes using orthogonality of sines:

$$\int_0^H f_0(y)\sin(m\pi y/H)dy = \sum_{n=1}^{\infty}\int_0^H b_n \sinh(n\pi L/H)\sin(n\pi y/H)\sin(m\pi y/H)dy$$

$$= \frac{H}{2} b_m \sinh(m\pi L/H)$$

so

$$b_m = \frac{2}{H\sinh(m\pi L/H)}\int_0^H f_0(y)\sin(m\pi y/H)dy$$

Note:

- We see that even though the boundary conditions were not *completely* homogeneous, separation of variables works here too. This is because we could pick the variable with homogeneous boundary conditions as the one for the Fourier expansion.
- Plotting the solution shows that it seems to be the smoothest possible one that fits the boundary conditions. This is a generic property of Laplace's equation.
- The maximum of the function f is achieved on the boundary of the domain. This is in fact a general property of solutions of Laplace's equation in bounded domains called the *Weak Maximum Principle.* The *Strong Maximum Principle* further states that if f achieves a maximum within the domain, then the only way this can happen is for f to be constant.

Finally, and perhaps most importantly, we note that in all of these three examples, the solution was expanded as a Fourier series in the x (or y) variable, which had homogeneous boundary conditions. This is not surprising, because the Fourier 'modes' are in each case the eigenfunctions of the operator $\mathcal{L}_x = \partial^2/\partial x^2$. Which modes are needed (sines or cosines, or combinations thereof), and their basic wavenumber, depends only on that operator and on the boundary conditions applied – they are independent of the initial or boundary conditions applied to the other variable.

In addition note how in each of these examples we relied heavily on the orthogonality of the modes to apply either initial conditions or boundary conditions. This property is therefore another key to the success of the method of separation of variables.

2.3 Second order 2D linear PDEs (somewhat more complicated problems)

Lecture edited by Charlie, Dante and Yiqin

In this section, we now consider somewhat more complicated problems, in which some simple form of forcing is applied, either through the boundary conditions, or in the PDE itself. For the sake of simplicity, we will use examples that build on the diffusion equation, wave equation, and Laplace's equation we studied in the last lecture, though the principles described here are more generally applicable.

2.3.1 Non-homogeneous boundary conditions

When dealing with a problem where the function f has non-homogeneous boundary conditions, we cannot apply the method of separation of variables directly, but the trick to solve it is quite simple: find *any* function h that satisfies the boundary conditions (but it does not have to satisfy the PDE), and then write $f = u + h$. It is easy to check that the function u must now satisfy homogeneous boundary conditions. This changes the PDE (by adding a term that makes it non-homogenous), and sometimes it also changes the initial conditions, but the key is that the boundary conditions are now the right ones for separation of variables.

Example 1: Solve the diffusion equation (2.13) on the interval $(0, L)$ with boundary conditions $f(0) = 0$ and $f(L) = 1$, and initial conditions $f(x, 0) = H(x - L/2)$ (the Heaviside function).

Solution:

Proof. Begin by writing $u(x, t) = f(x, t) - x/L$, then the new problem written in terms of u is:

$$\begin{cases} u_t = \kappa u_{xx} \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = H(x - L/2) - x/L \end{cases} \tag{EX 1}$$

The boundary conditions are now homogeneous so we can use separation of variables as usual:

$$u(x,t) = A(x)B(t) \implies B_t = -\lambda\kappa B, \quad A_{xx} = -\lambda A$$

Following a similar method as (2.13):

$$B(t) = b_n e^{-\lambda\kappa t}, \quad A(x) = a_{n0}\cos\left(\sqrt{\lambda}x\right) + a_{n1}\sin\left(\sqrt{\lambda}x\right)$$

$$(\mathbf{BC}) \implies a_{n0} = 0, \quad \lambda = \frac{n^2\pi^2}{L^2}$$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-t/\tau_n}, \quad \tau_n \equiv \frac{1}{\lambda\kappa}$$

Applying initial conditions, and then projecting onto the sine basis:

$$u(x,0) = H(x - L/2) - x/L \equiv u_0(x)$$

$$\implies c_m = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

We now have everything to assemble the final solution $f(x,t) = (x/L) + u(x,t)$.

□

Lecture edited by Julian, Sean and Moein

Example 2: Solve the wave equation (2.42) on the interval $(0, L)$ with boundary conditions $f(0) = 0$ and $f(L) = \sin(\omega_F t)$, where ω_F is some forcing frequency, and initial conditions $f(x, 0) = \partial f/\partial t(x, 0) = 0$.

Solution:

We write f as a sum of two separate functions, u and h, and seek an h that satisfies our B.C.'s

$$f(x,t) = u(x,t) + h(x,t).$$
$$\text{Let} \quad h(x,t) = \sin(\omega_F t)\frac{x}{L}$$

With this construction $u(x, t)$ satisfies homogeneous Dirichlet boundary conditions ($u(x, t) = 0$ at both ends). The initial conditions for u are:

$$u(x,0) = f(x,0) - h(x,0) = f(x,0) = 0$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left.\frac{\partial f}{\partial t}\right|_{t=0} - \left.\frac{\partial h}{\partial t}\right|_{t=0} = -\omega_F\frac{x}{L}$$

Substituting $f(x, t)$ into the wave equation, we finally obtain an equation for u:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + \sin(\omega_F t)\omega_F^2\frac{x}{L} \tag{2.47}$$

In this second problem, we see that the PDE itself now becomes non-homogeneous, even though we fixed the boundary conditions, which brings the next question : how to solve non-homogeneous PDEs with homogeneous boundary conditions.

2.3.2 Non-homogeneous (forced) PDEs (general considerations)

Let's now consider a generic PDE of the form

$$\frac{\partial^2 f}{\partial x^2} = \mathcal{L}_t f + F(x,t) \tag{2.48}$$

where t here either represents time or another spatial variable, $\mathcal{L}_t f$ is some linear operator in t only, and $F(x, t)$ is a forcing (non-homogeneous) term that does not contain f. We assume that the boundary conditions in x are homogeneous. We see that this form can be used to represent the forced diffusion equation (with $\mathcal{L}_t f = \partial f/\partial t$), the forced wave equation (with $\mathcal{L}_t f = \partial^2 f/\partial t^2$), and the forced Laplace equation (which is really called the Poisson equation), with $t \to y$ and (with $\mathcal{L}_y f = -\partial^2 f/\partial y^2$).

To solve this equation, we remember that in the method of separation of variables applied to the equivalent *unforced* problem (where $F = 0$), the solution f was expanded as a Fourier series which satisfied the boundary conditions. So let us assume here by analogy (for now) that the actual solution $f(x, t)$ *can* be expanded as a Fourier Series in x, where the Fourier modes are chosen to be the eigenmodes of $\partial^2 f/\partial x^2$ that satisfy the homogeneous boundary conditions in x (these could be sines, cosines, or both).

To focus the mind, let's assume that the domain is once again $[0, L]$ with homogeneous Dirichlet conditions. Then we know that the Fourier series only involves sine modes, of the kind $\sin(n\pi x/L)$. The ansatz for f is therefore

$$f(x,t) = \sum_{n=0}^{\infty} c_n(t)\sin\left(\frac{n\pi x}{L}\right) \tag{2.49}$$

Let us now substitute this ansatz into the PDE, and use linearity. We obtain

$$\sum_{n=0}^{\infty} -\frac{n^2\pi^2}{L^2}c_n(t)\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=0}^{\infty}\mathcal{L}_t c_n(t)\sin\left(\frac{n\pi x}{L}\right) + F(x,t) \tag{2.50}$$

This would normally not be too helpful, but here once again we can use the orthogonality property of the Fourier modes, namely equation (2.44) to project this equation onto a single Fourier mode and simplify it greatly.

So, by multiplying (2.50) by $\sin(m\pi x/L)$ and taking the integral over the interval $[0, L]$, all the terms in the infinite sums disappear, leaving only

$$-\frac{L}{2}\frac{m^2\pi^2}{L^2}c_m(t) = \frac{L}{2}\mathcal{L}_t c_m(t) + \int_0^L F(x,t)\sin\left(\frac{m\pi x}{L}\right)dx \tag{2.51}$$

or equivalently

$$\mathcal{L}_t c_m(t) = -\frac{m^2\pi^2}{L^2}c_m(t) - F_m(t) \tag{2.52}$$

where

$$F_m(t) = \frac{2}{L}\int_0^L F(x,t)\sin\left(\frac{m\pi x}{L}\right)dx \tag{2.53}$$

To solve the problem, we then simply have to solve this relatively simple linear ODE for each of the functions $c_m(t)$!

Before we move on to some examples, let's discuss first what enabled us to so conveniently go from PDEs to ODEs, and identify a few underlying assumptions that were made (and swept under the carpet).

- We *assumed* that it is possible to expand the solution as a Fourier series, and that the series exists and converges. This turns out to be possible only because Fourier modes used form a *complete* basis for functions on the interval $[0, L]$.
- We relied *heavily* on the orthogonality relationship (2.44) to project (2.50) onto each Fourier mode, and obtain a set of ODEs from the original PDE. Behind the scene, these orthogonality relationships exist because the basis is not only complete, but it is also an orthogonal basis.

Isn't it *so amazingly* advantageous that the Fourier modes not only form a complete basis of all functions in $[0, L]$, but also an orthogonal basis? Shouldn't we worry that it we move to harder problems where the sines and cosines are no longer the eigenmodes of the problem, we may lose that advantage? As it turns out, none of these properties of Fourier modes are a mere coincidence, and similar properties will be found in a large class of linear 2nd order PDEs thanks to Sturm-Liouville theory. This will be the topic of the next lecture.

In the meantime, let's now do a few examples of forced linear PDEs.

2.3.3 The oscillating rope

The second example of non-homogeneous boundary conditions earlier in this lecture can be viewed as a problem of a rope tied on one end, and the other end is shaken up and down with frequency ω_F. Let us now finish the problem to see what the solution is.

- Solve the problem analytically
- What is the fundamental frequency ω_0 of the unforced problem? What happens when $\omega_F \gg \omega_0$? What happens when $\omega_F \ll \omega_0$?

Let's go back to the problem in equation (2.47). We already know the general form of $u(x,t)$ from 2.2.5 to be:

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

Now plugging $u(x,t)$ into the equation above:

$$\sum_{n=1}^{\infty}\frac{d^2C_n}{dt^2}\sin\left(\frac{n\pi x}{L}\right) = c^2\sum_{n=1}^{\infty}[-\frac{n^2\pi^2}{L^2}\sin(\frac{n\pi x}{L})C_n(t)] + \sin(\omega_F t)\omega_F^2\frac{x}{L}$$

From here, we can use orthogonality of sines to eliminate a majority of the terms in our infinite sums.

$$\int_0^L \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})dx$$
$$= \begin{cases} 0 \text{ if m} \neq n \\ \frac{L}{2} \text{ otherwise} \end{cases}$$

We therefore project the equation onto $\sin(m\pi x/L)$:

$$\frac{L}{2}\frac{d^2C_m}{dt^2} = -c^2\frac{L}{2}(\frac{m^2\pi^2}{L^2})C_m(t) + \sin(\omega_F t)\frac{\omega_F^2}{L}\int_0^L x\sin(\frac{m\pi x}{L})dx$$

We also define

$$f_m = \frac{2}{L}\int_0^L x\sin(\frac{m\pi x}{L}dx)$$

so

$$\frac{d^2C_m}{dt^2} = -\frac{m^2\pi^2c^2}{L^2}C_m(t) + \sin(\omega_F t)\frac{\omega_F^2}{L}f_m$$

This is of the form $g'' = -\omega_m^2 g + \alpha\sin(\omega_F t)$ which has the general solution $g(t) = a\cos(\omega_m t) + b\sin(\omega_m t) + K\sin(\omega_F t)$ where a and b are arbitrary constants to be fitted to the initial conditions, and K is part of the particular solution for $g(t)$. Let

$$g_{ps}(t) = K\sin(\omega_F t)$$

We plug that into our ODE $g'' = -\omega_m^2 g + \alpha\sin(\omega_F t)$

$$-\omega_F^2 K\sin(\omega_F t) = -\omega_m^2 K\sin(\omega_F t) + \alpha\sin(\omega_F t)$$
$$\rightarrow K = \frac{\alpha}{\omega_m^2 - \omega_F^2}\ \omega \neq \omega_F$$

Combining are particular solution and general solution we have

$$C_m(t) = a_m\cos(\omega_m t) + b_m\sin(\omega_m t) + \frac{\omega_F^2 f_m}{L}\frac{1}{\omega_m^2 - \omega_F^2}\sin(\omega_F t) \tag{2.54}$$
$$\omega_m = \frac{m\pi c}{L} \tag{2.55}$$

Enforcing our initial condition $u(x,0) = 0$ we determine $a_n = 0$, $\forall n$. Then applying our other initial condition $\frac{\partial u}{\partial t}|_{(x,0)} = \frac{-x\omega_F}{L}$

$$-\frac{x\omega_F}{L} = \sum_{n=1}^{\infty}\sin(\frac{n\pi x}{L})\frac{dC_n}{dt}|_{t=0} \tag{2.56}$$

Which we can project using orthogonality with sines.

$$-\int_0^L \frac{x\omega_F}{L}\sin(\frac{m\pi x}{L})dx = \frac{dC_m}{dt}|_{t=0}\frac{L}{2} \tag{2.57}$$
$$\frac{dC_m}{dt}|_{t=0} = -\frac{\omega_F}{L}f_m \tag{2.58}$$

Applying this initial condition to the expression for $C_m(t)$, we find that

$$\omega_m b_m + K_m\omega_F = -\frac{\omega_F}{L}f_m \tag{2.59}$$

Then solving for b_m

$$b_m = -\frac{\omega_F}{\omega_m}(\frac{f_m}{L} + K_m)$$

Altogether we end up with a solution for $f = u + h$

$$f(x,t) = \frac{x}{L}\sin(\omega_F t) + \sum_{n=1}^{\infty}\sin(\frac{n\pi x}{L})(b_n \sin(\omega_n t) + K_n \sin(\omega_F t)) \tag{2.60}$$

Behavior of the Solution:

- **When $\omega_F \gg \omega_0$:** The forcing frequency is much higher than the fundamental frequency of the system. In this case, the rope cannot follow the fast oscillations of the shaking end effectively, and the amplitude of the response will be small because the system is not in resonance.
- **When $\omega_F \ll \omega_0$:** The forcing frequency is much lower than the fundamental frequency. Here, the rope can respond slowly to the shaking, but the oscillations will be out of sync with the natural frequency, leading to low-amplitude motions far from the driven end.
- **Resonance:** If ω_F approaches one of the natural frequencies $\omega_n = \frac{n\pi c}{L}$, resonance occurs. At resonance, the amplitude of oscillation becomes large, and the rope oscillates in a mode corresponding to n.

2.3.4 The forced diffusion equation

A pub in England rings last orders at 11pm, at which point people start leaving to go home. They are all 'locals' which means they live in the same 1D street as the pub. The street has length L and we assume the pub is in the middle of it. The people are quite drunk, and walk around randomly in the street, but don't leave it. They can't find their homes or their keys, which means they end up staying in the street for a long time. We model this problem mathematically using the following equations:

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} + S(x,t) \tag{2.61}$$

$$p(x,0) = 0 \tag{2.62}$$

$$\frac{\partial p}{\partial x} = 0 \text{ at } x = 0, L \tag{2.63}$$

where p is the probability density of drunk people, $S(x,t)$ is the 'source' of drunk people per unit time coming into the street at position x. We let $t = 0$ corresponds to 11pm.

To model the exit of the pub, we will set

$$S(x,t) = S_0\delta(x - L/2)e^{-t/\tau} \text{ for } t > 0 \tag{2.64}$$

where τ is some characteristic timescale.

- Use dimensional analysis to find the characteristic diffusion timescale of the problem in the absence of forcing, τ_D
- Solve this problem using the method discussed in this section.
- Plot the solution numerically to find what happens when $\tau \ll \tau_D$? What happens when $\tau \gg \tau_D$?

Solution:

- **Dimensional Analysis:** The diffusion coefficient D has units of [length]^2/[time], and the characteristic length scale of the problem is L. We can therefore construct a characteristic timescale as

$$\tau_D = \frac{L^2}{D}$$

- **Separation of Variables:** Now, consider the forced diffusion equation:

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} + S(x,t)$$

We assume a solution of the form:

$$p(x,t) = X(x)T(t)$$

Substituting this into the homogeneous part of the equation (i.e., without the source term $S(x,t)$):

$$X(x)\frac{dT(t)}{dt} = DT(t)\frac{d^2X(x)}{dx^2}$$

Dividing both sides by $X(x)T(t)$, we separate the variables:

$$\frac{1}{T(t)}\frac{dT(t)}{dt} = D\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\lambda$$

where λ is a separation constant. This leads to two ordinary differential equations (ODEs): - For the time part:

$$\frac{dT(t)}{dt} = -\lambda T(t)$$

- For the space part:

$$\frac{d^2X(x)}{dx^2} = -\frac{\lambda}{D}X(x)$$

Solving the space part, we obtain the general solution:

$$X(x) = A\cos\left(\frac{\sqrt{\lambda}}{\sqrt{D}}x\right) + B\sin\left(\frac{\sqrt{\lambda}}{\sqrt{D}}x\right)$$

Given that the boundary conditions are Neumann ($\frac{\partial p}{\partial x} = 0$ at $x = 0$ and $x = L$), we find that $B = 0$, and the eigenvalues are:

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

So the spatial part of the solution is:

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

For the time part:

$$T_n(t) = e^{-\lambda_n t} = e^{-t/\tau_n}$$

where we now use $\tau_n = \frac{L^2}{Dn^2\pi^2}$ as the diffusion timescale for mode n.

- **General Solution:** The solution for $p(x,t)$ is a sum over all eigenmodes:

$$p(x,t) = \sum_{n=0}^{\infty} c_n(t)\cos\left(\frac{n\pi x}{L}\right)$$

Substituting this back into the forced diffusion equation:

$$\sum_{n=0}^{\infty}\frac{dc_n(t)}{dt}\cos\left(\frac{n\pi x}{L}\right) = \sum_{n=0}^{\infty} -\frac{1}{\tau_n}c_n(t)\cos\left(\frac{n\pi x}{L}\right) + S(x,t)$$

- **Orthogonality and Projection:** Using the orthogonality of the cosine modes, we project onto the m-th mode by multiplying by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating over the interval $[0, L]$. This leads to the following ODE for the time-dependent coefficients $c_m(t)$:

$$\frac{dc_m(t)}{dt} = -\frac{1}{\tau_m}c_m(t) + \frac{2}{L}\int_0^L S(x,t)\cos\left(\frac{m\pi x}{L}\right)dx \text{ if } m > 0$$

and

$$\frac{dc_0(t)}{dt} = \frac{1}{L}\int_0^L S(x,t)dx \text{ if } m = 0$$

- **Solving the ODE:** The source term $S(x,t)$ is given by $S_0\delta(x-L/2)e^{-t/\tau}$, so we compute:

$$\frac{2}{L}\int_0^L S_0\delta(x-L/2)e^{-t/\tau}\cos\left(\frac{m\pi x}{L}\right)dx = \frac{2S_0}{L}e^{-t/\tau}\cos\left(\frac{m\pi}{2}\right)$$

  Thus, the ODE for $m>0$ becomes:

$$\frac{dc_m(t)}{dt} = -\frac{1}{\tau_m}c_m(t) + \frac{2S_0}{L}e^{-t/\tau}\cos\left(\frac{m\pi}{2}\right)$$

  This is a first-order linear ODE and can be solved using an integrating factor:

$$c_m(t) = e^{-t/\tau_m}c_m(0) + \frac{2S_0}{L}\cos\left(\frac{m\pi}{2}\right)e^{-t/\tau_m}\int_0^t e^{-t'/\tau+t'/\tau_m}dt'$$

  Applying the initial condition $c_m(0)=0$

$$c_m(t) = \frac{2S_0}{L}\cos\left(\frac{m\pi}{2}\right)\frac{e^{-t/\tau}-e^{-t/\tau_m}}{\frac{1}{\tau_m}-\frac{1}{\tau}}$$

- **Finding b_0 from the Zeroth Mode Projection:** For the zeroth mode $m=0$,

$$\frac{dc_0(t)}{dt} = \frac{S_0}{L}e^{-t/\tau}$$

  Solving this ODE using $c_0(0)=0$:

$$c_0(t) = \frac{S_0\tau}{L}\left(1-e^{-t/\tau}\right)$$

- **Final Solution:** The final solution for $p(x,t)$ is:

$$p(x,t) = c_0(t) + \frac{2S_0}{L}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right)}{D\left(\frac{n\pi}{L}\right)^2-\frac{1}{\tau}}\left(e^{-t/\tau}-e^{-t/\tau_n}\right)\cos\left(\frac{n\pi x}{L}\right)$$

- Plotting the solution: By plotting the final solution numerically, we see how large and small diffusion time scale can vary the solution.

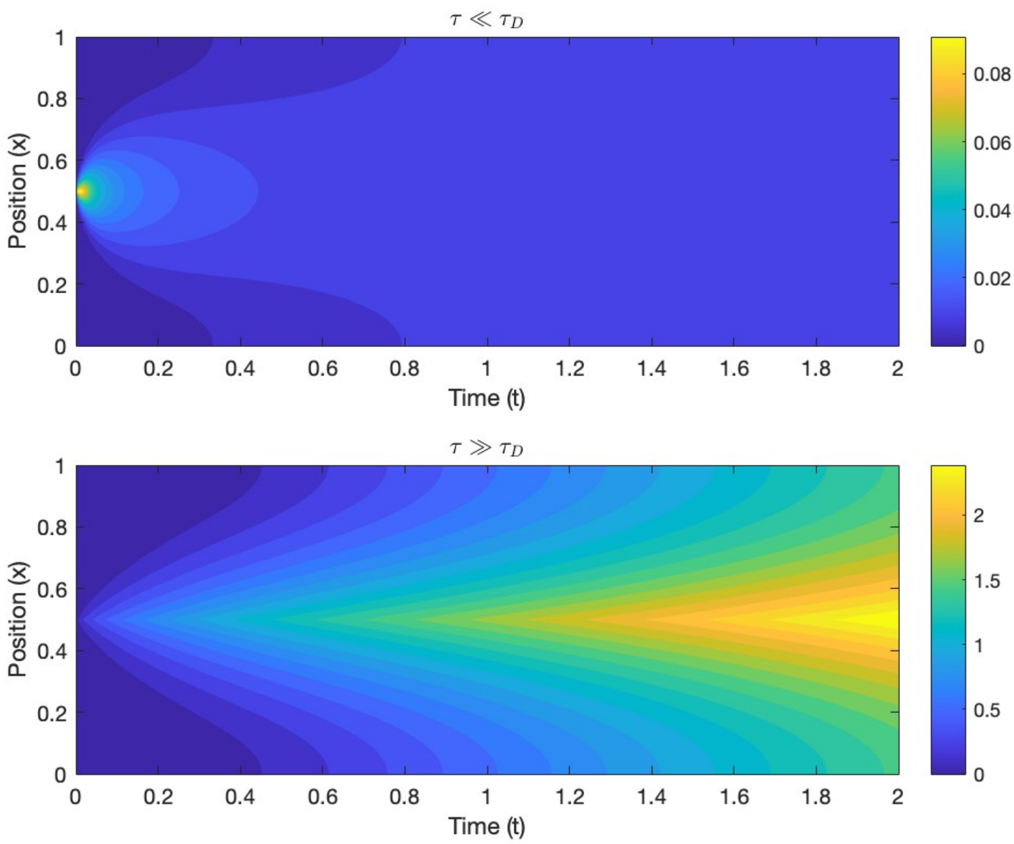

Figure 2.1: Comparison of diffusion for $\tau \ll \tau_D$ and $\tau \gg \tau_D$

In the last 2 lectures, we covered simple 'canonical' second order 2D PDEs (diffusion, wave, Poisson equations). In all of these cases, the coefficients of the principal part were constant, which resulted in eigenmodes of the spatial problem that are simple Fourier (sine, cosine) functions. In general, however, the coefficients are not always constant and these second-order 2D linear PDEs have the more universal form (2.3). Let's now study how to deal with these problems.

2.4 Second-order 2D linear PDEs (non-constant coefficients)

Lecture edited by Janice, Kevin and Dante.

There are many common situations in which the coefficients of the principal part of the PDE can depend on the independent variables:

- For the standard diffusion and wave equations in Cartesian coordinates, this happens when the wave speed and the diffusion coefficient vary with x, t.
- The diffusion, wave and Laplace equations also acquire non-constant coefficients when moving to different coordinate systems that are more appropriate for the geometry of the problem.

2.4.1 The standard equations in other coordinate systems

In order to have a chance at using separation of variables to solve a problem, the domain boundaries should be aligned with the coordinate system. This means that to solve a PDE in a disk, it is better to use polar coordinates; to solve it in a cylinder or a sphere, it is better to use cylindrical or spherical coordinate systems; other coordinate systems also exist and/or can be constructed to deal with more complicated shapes.

To find the correct expression for the wave, diffusion and Laplace's equation (or any other PDE) in other coordinate systems, it is very important to always keep in mind the *principle of coordinate independence*. That is, while *you* may prefer to use a particular coordinate system to model a certain physical phenomenon, that phenomenon exists and is the same regardless of the coordinate system it is expressed in. For this reason, it is always better to write PDEs first using differential operators that are universal, and then express them in the coordinate system of your choice once you have selected it.

These universal differential operators are ∇, $\nabla\cdot$ and $\nabla\times$ at first order, and the combination of $\nabla\cdot$ and ∇ applied to a scalar is the so-called Laplacian operator

$$\nabla^2 f = \nabla \cdot (\nabla f) \tag{2.65}$$

Using these operators,

- The universal expression for the diffusion equation (with constant diffusion coefficient) is:

$$\frac{\partial f}{\partial t} = D\nabla^2 f \tag{2.66}$$

- The universal expression for the wave equation (with constant wave speed c is:

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f \tag{2.67}$$

- The universal expression for Laplace's equation is:

$$\nabla^2 f = 0 \tag{2.68}$$

To express these equations in a given coordinate system, see, e.g. the NRL plasma formulary (for cylindrical adn spherical coordinates) and Batchelor's book (for general coordinates), to find the expression of the Laplacian operator in that coordinate system.

Example 1: Laplace's equation in polar coordinates is:

Solution:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0$$
$$r^2\frac{\partial^2 f}{\partial r^2} + r\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \theta^2} = 0$$

Example 2: The spherically-symmetric wave equation is:

Solution:

$$\begin{aligned}\frac{\partial^2 f}{\partial t^2} &= c^2\nabla^2 f \\ &= c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)\right] \\ &= c^2\left[\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}\right]\end{aligned}$$

We see in these examples that even the 'standard' PDEs now have coefficients that depend on space.

2.4.2 Separation of variables

The general form of the homogeneous diffusion, wave and Laplace equations in other coordinate systems is still

$$\mathcal{L}_t f = \mathcal{L}_x f \tag{2.69}$$

where $\mathcal{L}_x$ is a second-order linear differential operator that only involves a spatial coordinate x, $\mathcal{L}_t$ is a first or second-order differential operator that only involves t, and where $t \to y$ for the Poisson equation. The main difference with the previous sections is that these operators now have coefficients that depend on the independent variables. Many PDEs arising from physical systems can be written in such a form, so what follows does have a lot of applications.

As we saw in Lecture 2, this equation has separable solutions of the form $f(x,t) = A(x)B(t)$ provided we can find temporal and spatial eigenmodes of the problem such that

$$\mathcal{L}_x A = \lambda A, \text{ and } \mathcal{L}_t B = \lambda B. \tag{2.70}$$

Everything therefore hangs in the existence of eigensolutions to the spatial part of the problem, that satisfy the boundary conditions. This existence, and the properties of the eigensolutions, are explicitly given by Sturm-Liouville theory.

Let us first explicitly write, in all generality,

$$\mathcal{L}_x A = a(x)\frac{d^2A}{dx^2} + b(x)\frac{dA}{dx} + c(x)A. \tag{2.71}$$

Note that there is no term without A, because we had assumed the problem is homogeneous. Also note that since A is only a function of x, we can now use regular derivatives.

Let's multiply this equation by the function $r(x) = p(x)/a(x)$, where

$$p(x) = \exp\left(\int \frac{b(x)}{a(x)}dx\right). \tag{2.72}$$

Note that for $p(x)$ to exist, $b(x)/a(x)$ must be integrable, so there are limitations to this method if that is not the case. If this is reminiscent of the integrating factor method for 1st order ODEs, that is not a coincidence! We get

$$r(x)\mathcal{L}_x A = p(x)\frac{d^2A}{dx^2} + \frac{p(x)}{a(x)}b(x)\frac{dA}{dx} + r(x)c(x)A \tag{2.73}$$

Noting that

$$p'(x) = \frac{b(x)}{a(x)} p(x) \tag{2.74}$$

we see that

$$r(x)\mathcal{L}_x A = p(x)\frac{d^2A}{dx^2} + p'(x)\frac{dA}{dx} + r(x)c(x)A = \frac{d}{dx}\left[p(x)\frac{dA}{dx}\right] + r(x)c(x)A \tag{2.75}$$

The eigenvalue problem for A then becomes

$$\frac{d}{dx}\left[p(x)\frac{dA}{dx}\right] + q(x)A = \lambda r(x)A \tag{2.76}$$

where $q(x) = r(x)c(x)$. As we shall see now, this is called a Sturm-Liouville eigenvalue problem, and so we have just shown that separable linear second-order 2D homogeneous PDEs *almost* always reduce to a Sturm Liouville problem (the only exception being when $p(x)$ does not exist because $b(x)/a(x)$ is not integrable in the domain considered).

Examples:

- Consider Laplace's equation in spherical coordinates. After assuming that the solutions are proportional to $\sin(\phi)$, separate the remaining variables. Show that the eigenvalue equations in r and θ are both in the form (2.76).

  Solution:
  The Laplacian in spherical coordinates is:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = 0 \tag{2.77}$$

  We assume separation of variables with the solution proportional to $\sin(\phi)$:

$$f(r,\theta,\phi) = A(r)B(\theta)\sin(\phi) \tag{2.78}$$

  Substituting in the Laplacian:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)B\sin\phi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial B}{\partial\theta}\right)A\sin\phi - \frac{AB}{r^2\sin^2\theta}\sin\phi = 0 \tag{2.79}$$

  Simplifying and re-arranging we get:

$$\frac{1}{A}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right) = \frac{1}{\sin^2\theta} - \frac{1}{B}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial B}{\partial\theta}\right) = \lambda \tag{2.80}$$

  This gives us the eigenfunctions:

$$\frac{d}{dr}\left(r^2\frac{dA}{dr}\right) = \lambda A \quad \text{and} \quad \frac{d}{d\theta}\left(\sin\theta\frac{dB}{d\theta}\right) - \frac{B}{\sin\theta} = -\lambda\sin\theta B \tag{2.81}$$

- Consider the equation:

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} + \lambda r^2 u = 0 \tag{2.82}$$

  where ν is a constant. Put it in the form (2.76).

  Solution:
  We start by dividing by r:

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} + \lambda r u = 0 \tag{2.83}$$

$$\rightarrow \frac{d}{dr}\left[r\frac{du}{dr}\right] = -\lambda r u \tag{2.84}$$

We now begin our exploration of Sturm-Liouville theory by formally defining what a Sturm-Liouville problem is.

2.4.3 Sturm-Liouville problems

The eigenvalue problem

$$\mathcal{L}(u) = \frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = -\lambda w(x)u \tag{2.85}$$

on the interval (x_a, x_b), with homogeneous boundary conditions

$$\alpha_a u(x_a) + \beta_a u'(x_a) = 0 \tag{2.86}$$
$$\alpha_b u(x_b) + \beta_b u'(x_b) = 0 \tag{2.87}$$

is called a Sturm-Liouville problem provided

- $p(x)$, $p'(x)$, $q(x)$ and $w(x)$ are defined and continuous in (x_a, x_b)
- $p(x) > 0$ and $w(x) > 0$ in (x_a, x_b)
- $|\alpha_a| + |\beta_a| > 0$,$|\alpha_b| + |\beta_b| > 0$

If $p(x)$ or $w(x)$ vanish at one of the boundaries, or if the domain is unbounded, the problem is called a *singular* Sturm-Liouville problem. Otherwise the problem is called *regular*. Also note that it is also possible to consider periodic boundary conditions, such that

$$u(x_a) = u(x_b) \text{ and } u'(x_a) = u'(x_b) \tag{2.88}$$

Problems with periodic boundary conditions have very similar properties to regular Sturm-Liouville problems (as long as $p(x) > 0$, $w(x) > 0$ on $[a, b]$).

The function $w(x)$ is often called *the weight function* of the problem, and we will see shortly why. Note also that we have redefined the sign of λ from the previous section, to be consistent with the standard definitions used in Sturm-Liouville theory.

Example 1: Consider the equation and boundary conditions:

$$\frac{d^2u}{dx^2} + \lambda u = 0 \tag{2.89}$$
$$u(0) = 0, u(1) = 0 \tag{2.90}$$

Identify if it is a Sturm-Liouville problem, and if yes, what type it is.

Solution:

This is a regular Sturm-Liouville problem with:

$$p(x) = 1 \quad \text{and} \quad q(x) = 0 \quad \text{and} \quad w(x) = 1 \tag{2.91}$$

Note that neither $p(x)$ nor $w(x)$ vanish at the boundaries, and the domain is bounded.

Example 2: Consider the Bessel equation and boundary conditions:

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} + \lambda r^2 u = 0 \tag{2.92}$$
$$|u(0)| < +\infty, u(R) = 0 \tag{2.93}$$

In the previous lecture, we already put this equation in the relevant form. Identify if it is a Sturm-Liouville problem, and if yes, what type it is.

Solution: After putting the equation in the form (2.84), we see that this is a singular Sturm-Liouville problem with:

$$p(r) = r, q(r) = 0, w(r) = r \tag{2.94}$$

Note that $p(r)$ and $w(r)$ vanish at $r = 0$.

As we shall now demonstrate, Sturm-Liouville problems are equivalent to the 'real symmetric matrices' of linear algebra, and therefore have similar properties when it comes to their eigenvalues and eigenfunctions.

2.4.4 Properties of Sturm-Liouville problems

(1) Symmetry of the operator.
It is easy to show that the operator $\mathcal{L}$ is symmetric, where symmetry is defined here as the property that

$$\int_{x_a}^{x_b} [u\mathcal{L}(v) - v\mathcal{L}(u)]\, dx = 0 \tag{2.95}$$

for any two functions u and v satisfying the boundary conditions.

Proof. Begin by expanding the Sturm-Liouville operator inside the integral and simplifying,

$$\int_{x_a}^{x_b} [u\mathcal{L}(v) - v\mathcal{L}(u)]\, dx = \int_{x_a}^{x_b} u\left(\frac{d}{dx}\left(p\frac{dv}{dx}\right) + qv\right) - v\left(\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right) dx$$
$$= \int_{x_a}^{x_b} u\frac{d}{dx}\left(p\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p\frac{du}{dx}\right) dx$$

Then proceed by using Integration by parts and simplifying,

$$= up\frac{dv}{dx}\Big|_{x_a}^{x_b} - vp\frac{du}{dx}\Big|_{x_a}^{x_b} - \int_{x_a}^{x_b}\left(p\frac{du}{dx}\frac{dv}{dx} - p\frac{du}{dx}\frac{dv}{dx}\right) dx$$
$$= u(x_b)p(x_b)\frac{dv}{dx}(x_b) - v(x_b)p(x_b)\frac{du}{dx}(x_b) - u(x_a)p(x_a)\frac{dv}{dx}(x_a) + v(x_a)p(x_a)\frac{du}{dx}(x_a)$$

Since both u and v satisfy the boundary conditions at x_a and x_b we can observe the following relationship between each function and its first derivative at each boundary.:

$$\text{(BC)} \implies v'(x_i) = -\frac{\alpha_i}{\beta_i}v(x_i)$$

This allows us to complete the proof using substitution,

$$= u(x_b)p(x_b)\left(-\frac{\alpha_b}{\beta_b}v(x_b)\right) - v(x_b)p(x_b)\left(-\frac{\alpha_b}{\beta_b}u(x_b)\right) - u(x_a)p(x_a)\left(-\frac{\alpha_a}{\beta_a}v(x_a)\right) + v(x_a)p(x_a)\left(-\frac{\alpha_a}{\beta_a}u(x_a)\right)$$
$$= 0, \quad \text{The Sturm-Liouville operator is symmetric}$$

□

(2) Orthogonality of the eigenfunctions.
The eigenfunctions of a Sturm-Liouville problem are orthogonal with respect to the inner product

$$\langle u, v\rangle = \int_{x_a}^{x_b} u(x)v(x)w(x)dx \tag{2.96}$$

Proof: Assume $v_n(x), v_m(x)$ are both eigenfunctions, then

$$\mathcal{L}v_n(x) = -\lambda_n w(x)v_n(x) \tag{2.97}$$
$$\mathcal{L}v_m(x) = -\lambda_m w(x)v_m(x) \tag{2.98}$$

Because the operator is symmetric, we have

$$\int_{x_a}^{x_b} (v_m\mathcal{L}v_n(x) - v_n\mathcal{L}v_m(x))\, dx = 0 \tag{2.99}$$

$$\to \int_{x_a}^{x_b} [-\lambda_n w(x)v_n(x)v_m(x) + \lambda_m w(x)v_n(x)v_m(x)] = 0 \tag{2.100}$$

$$\int_{x_a}^{x_b} (-\lambda_n + \lambda_m)\, w(x)v_n(x)v_m(x)dx = 0 \tag{2.101}$$

$$(-\lambda_n + \lambda_m)\langle v_n, v_m\rangle = 0 \tag{2.102}$$

$\Rightarrow$ Eigenfunctions associated with different eigenvalues are orthogonal with respect to this inner product.

(3) The eigenvalues of Sturm-Liouville problems are real

Proof. Assume $u(x)$, λ are an eigenfunction eigenvalue pair of a Sturm-Liouville operator $\mathcal{L}$. Begin by supposing that $p, q, w \in \mathbb{R}$ and then taking the complex conjugate of $\mathcal{L}(u)$ (denoted here with an overbar)

$$\overline{\mathcal{L}(u)} = \overline{\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)} + \overline{qu} = -\overline{\lambda w u}$$
$$= \frac{d}{dx}\left(p(x)\frac{d\overline{u}}{dx}\right) + q\overline{u} = \mathcal{L}(\overline{u}) = -\overline{\lambda} w \overline{u}$$

This implies that $\overline{u}$ is an eigenfunction of $\mathcal{L}$ with corresponding eigenvalue $\overline{\lambda}$. We have then that both u and $\overline{u}$ are eigenfunctions of $\mathcal{L}$. Since $\langle u, \overline{u}\rangle = \int_{x_a}^{x_b} |u|^2 w dx \neq 0$ we have that, u and $\overline{u}$ are not orthogonal. Since all eigenfunctions must be orthogonal, it must be the case that $u = \overline{u}$ and $\lambda = \overline{\lambda}$. Therefore, $\lambda \in \mathbb{R}$ and all eigenvalues are real.

□

The following proofs being somewhat more involved, we will skip them. However, note that while all of the properties so far applied to any Sturm-Liouville problem, the next ones only work for regular Sturm-Liouville problems.

(4) The eigenvalues of regular Sturm-Liouville problems are simple.
In practice, this means that if two functions have the same eigenvalue, then these two functions are linearly dependent.

(5) The set of all eigenvalues of a regular Sturm-Liouville problem form an unbounded, strictly monotone sequence.
In other words, the set of all eigenvalues can be ordered as

$$\lambda_0 < \lambda_1 < \lambda_2 < \ldots \tag{2.103}$$

with $\lim_{n\to\infty} \lambda_n = +\infty$. The quantity λ_0 is called the *principal eigenvalue.*

(6) The n-th eigenfunction of a regular Sturm-Liouville problem (i.e. the eigenfunction corresponding to λ_n has exactly n zeros in (x_a, x_b)

(7) The set of eigenfunctions of a regular Sturm-Liouville problem forms a complete basis for all functions on $[x_a, x_b]$. Furthermore, it is possible to construct this basis so that

- All of the eigenfunctions are real
- The basis is orthogonal (i.e. the eigenfunctions are all mutually orthogonal to each other)

This final property is obviously the most interesting one in the context of solving linear PDEs, because it allows us to generalize the concept of Fourier Series to other families of function. In particular, we now know that if the eigenfunctions of $\mathcal{L}$ are denoted as the family $\{v_n(x)\}$, then *any* function $u(x)$ defined on the interval $[x_a, x_b]$ can be written as

$$u(x) = \sum_{n=0}^{n=\infty} c_n v_n(x) \tag{2.104}$$

where, by orthogonality, the coefficients c_n are given by

$$c_n = \frac{\langle u, v_n\rangle}{\langle v_n, v_n\rangle} = \frac{\int_{x_a}^{x_b} u(x)v_n(x)w(x)dx}{\int_{x_a}^{x_b} v_n^2(x)w(x)dx} \tag{2.105}$$

2.4.5 Famous Examples of Sturm-Liouville problems

Example 1: The Fourier functions.
Consider the problem

$$\frac{d^2u}{dx^2} = -\lambda u$$
$$u(0) = u(L) = 0 \tag{2.106}$$

Show that it is a regular SL problem, and check that it satisfies all of the properties outlined in the previous section.

Solution: This PDE can be shown to be a Sturm-Liouville problem with the following form:

$$\frac{d}{dx}\left(\frac{du}{dx}\right) = -\lambda u, \implies p(x) = 1, \quad q(x) = 0, \quad w(x) = 1$$

The functions p, q, w satisfy the requirements for a regular Sturm-Liouville problem. Furthermore the boundary conditions can also be shown to satisfy the Sturm-Liouville condition with $\alpha_a = 1, \beta_b = 0, \alpha_b = 1$, and $\beta_b = 0$ where $x_a = 0$ and $x_b = L$. Therefore this is a regular Sturm-Liouville problem.

Solutions to this problem are

$$u_n(x) = \sin\left(\frac{n\pi x}{L}\right), \lambda_n = \frac{n^2\pi^2}{L^2} \tag{2.107}$$

Indeed, we know from the orthogonality of sines that the eigenfunctions are mutually orthogonal, with

$$\int_0^L u_n(x)u_m(x)dx = \frac{L}{2}\delta_{m,n} \tag{2.108}$$

We see that the eigenvalues are real, form a strictly increasing sequence. We have that the $sin(n\pi x/L)$ function indeed has n zeros on $(0, L)$. Finally, we also know from the theory of Fourier series, that these eigenfunctions indeed form a complete basis for the interval $[0, L]$.

Example 2: The Bessel Functions
Consider the problem

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} + \lambda r^2 u = 0$$
$$|u(0)| < +\infty, u(R) = 0 \tag{2.109}$$

Check that it satisfies the first three properties outlined in the previous section. What is the relevant orthogonality condition? Which of the other properties does it satisfy? Which does it not satisfy? (You will have to consult the *Handbook of Mathematical Functions* for the answer to some of these questions).

Solution:

We have already shown that this is a singular Sturm Liouville problem with $p(r) = r$, $w(r) = r$. We therefore construct the corresponding inner product as:

$$\langle u(r), v(r)\rangle = \int_0^R u(r)v(r)rdr$$

By setting $\lambda r^2 = x^2$, we can transform the equation into

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + x^2 u = 0$$
$$|u(x=0)| < +\infty, u(x = \sqrt{\lambda}R) = 0 \tag{2.110}$$

This is the general equation for a Bessel function of order 0, whose general solution is

$$u(x) = c_1 J_0(x) + c_2 Y_0(x)$$

For the Dirichlet conditions provided the singular condition holds if $c_2 = 0$ and the boundary condition is satisfied if $J_0(\sqrt{\lambda}R) = 0$. Let z_n be the zeros of the Bessel equation of the first kind of order 0. We know that the eigenvalues are:

$$\lambda_n = \frac{z_n^2}{R^2}, \quad n = 1, 2, 3, ...$$

so we see that they indeed form a sequence of strictly increasing numbers. The eigenfunctions are:

$$\phi_n(r) = J_0(\frac{z_n}{R}r)$$

We can check (cf. equation 11.4.5 in Abramowitz and Stegun) that they indeed satisfy the orthogonality condition

$$\int_0^R \phi_n(r)\phi_m(r) r dr = 0 \tag{2.111}$$

if $m \neq n$.

For a function

$$g(r) = \sum_{n=1}^{\infty} c_n J_v(\frac{z_n}{R}r), \quad c_n = \frac{\langle g, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_0^R g(r) J_0(\frac{z_n}{R}r) r dr}{\int_0^R \left(J_0(\frac{z_n}{R}r)\right)^2 r dr}$$

Thus all conditions (4) - (7) are satisfied.

Note that completeness of a basis is difficult to show in general in the (infinite) space of functions. As it turns out, it can be shown that this set of Bessel functions is a complete basis for all functions on the interval $[0, R]$.

We therefore see that even though the second example is a singular Sturm-Liouville problem, it still satisfies many of the same properties as those of a regular Sturm-Liouville problem (and in particular, the completeness of the basis). The reason behind this is that the singularity of this equation is not a 'bad' singularity. In technical terms the point $x = 0$ has a *regular singularity* (we will see more about those later), which are not as bad as real singularities. A physical reason one could invoke to understand why this problem effectively behaves as a regular problem is that it was *derived* from a regular problem simply through a change of variables. Therefore, we know that this singular is just a coordinate singularity and there is nothing physically singular about the problem.

Lecture edited by Henry, Alexandra and Jeremy.

Let us now put together everything that we learned so far to solve a few applied 2D PDEs.

2.4.6 Cooking an egg

Let's consider a spherical egg of radius $R = 2$cm. It comes out of the fridge with a uniform temperature of 4°C at time $t = 0$ and gets cooked in boiling water (100°C) for some time until it is cooked. It is considered fully cooked when the temperature at the center of the egg reaches 70°C. You may assume that the diffusion coefficient of temperature in egg whites and egg yolks are the same, and approximately equal to $D = 0.002\text{cm}^2/\text{s}$.

- Using dimensional analysis, what is the characteristic timescale associated with this problem?
- Using separation of variables, solve the problem and show that it can be written in the form

$$T(r,t) = 100 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} j_0\left(\frac{n\pi r}{R}\right) \tag{2.112}$$

  where j_0 is the Spherical Bessel function of order 0. Note that

$$j_0(x) = \frac{\sin(x)}{x} \tag{2.113}$$

- What is the orthogonality relation for the j_0 functions? Use it in conjunction with the initial conditions to find the a_n coefficients.
- Plot the solution in Matlab. How long does it take to cook the egg?
- What went wrong with the dimensional analysis?

Solution:

To find the characteristic timescale associated with the problem, we consider the parameters and their associated dimensions, and isolate a characteristic timescale T_c in terms of the parameters:

$$[D] = \frac{\text{length}^2}{\text{time}} \implies T_c = \frac{R^2}{D}$$
$$[R] = \text{length}$$
$$[T_0] = \text{Temperature}$$

Thus we have the characteristic timescale $T_c = \frac{R^2}{D}$, which is approximately 33 minutes.

Next, we use separation of variables to solve the problem.
First we consider our problem:

$$\frac{\partial T}{\partial t} = \nabla^2 T$$
$$T(r,0) = 4$$
$$T(R,t) = 100$$

Now - because we assumed a spherical egg, and given that our boundary conditions and initial conditions are spherically symmetric, our solution will only depend on r so - let's express our problem in spherical coordinates:

$$\frac{\partial T}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$$

Here, we use the auxiliary function that satisfies the boundary conditions to obtain homogeneous boundary conditions.

$$h(r) = 100$$

Now, we redefine the problem such that $T(r,t) = h(r) + u(r,t)$:

$$\begin{aligned}
\frac{\partial(h+u)}{\partial t} &= D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial(u+h)}{\partial r}\right)\\
\implies \frac{\partial u}{\partial r} &= D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)\\
u(R,t) &= 0\\
u(r,0) &= T(r,0) - 100 = -96
\end{aligned} \tag{2.114}$$

Now to solve this problem with homogeneous boundary conditions, let us assume the solution is of the form:

$$u(r,t) = A(r)B(t)$$

then with substitution into the PDE, we obtain:

$$\frac{1}{B}\frac{\partial B}{\partial t} = \frac{D}{r^2}\frac{1}{A}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)$$

and thus have two ODEs:

$$\begin{aligned}
\frac{\partial B}{\partial t} &= -\lambda B\\
\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right) &= -\lambda\frac{r^2}{D}A
\end{aligned}$$

The solution to the first ODE is:

$$B(t) = Ce^{-\lambda t} \tag{2.115}$$

with C a constant. Now we consider the second ODE. We may observe that is already in Sturm-Liouville form with

$$\begin{aligned}
p(r) &= r^2\\
q(r) &= 0\\
w(r) &= r^2
\end{aligned}$$

we consider our interval $r \in [0,R]$, and observe that at $r = 0$, $p(r) = 0$, and thus we have a singular Sturm-Liouville problem [1].

Now to find the solution to the ODE, we expand and obtain:

$$r^2\frac{\partial^2 A}{\partial r^2} + 2r\frac{\partial A}{\partial r} + \frac{\lambda}{D}r^2 A = 0 \tag{2.116}$$

We may now observe this equation is similar to the *spherical* Bessel equation of degree α:

$$x^2\frac{\partial^2 y}{\partial x^2} + 2x\frac{\partial y}{\partial x} + (x^2-\alpha^2)y = 0$$

In order to get these two equations to match, we first set $\alpha = 0$:

$$x^2\frac{\partial^2 y}{\partial x^2} + 2x\frac{\partial y}{\partial x} + x^2 y = 0$$

We next make the nonlinear substitution:

$$x^2 = \frac{\lambda}{D}r^2 \implies r = \sqrt{\frac{D}{\lambda}}x$$

[1] Note: This is not a true, geometric singularity where e.g. $p(r) = 0$ within the interval or where the domain is unbounded. This is merely a singularity induced by our choice of coordinates, i.e. a *coordinate singularity* where $p(r)$ and $w(r)$ vanish on the boundaries of the interval.

As mentioned previously in these course notes, the presence of coordinate singularities do not compromise this SL problem's ability to maintain most of the same desired properties as regular SL problems, such as its eigenfunctions forming a complete basis that we may use to represent its solution. For the same reason using polar coordinates for a map of the earth does not magically spawn a black hole in the North Pole.

Substituting this gives:

$$\begin{aligned}
\left(\frac{D}{\lambda}x^2\right)\frac{\partial^2 A}{\partial(\frac{D}{\lambda}x^2)} + 2\left(\sqrt{\frac{D}{\lambda}}x\right)\frac{\partial A}{\partial(\sqrt{\frac{D}{\lambda}}x)} + \frac{\lambda}{D}\left(\frac{D}{\lambda}x^2\right)A = 0 \\
\implies x^2\frac{\partial^2 A}{\partial x^2} + 2x\frac{\partial A}{\partial x} + x^2 A = 0
\end{aligned}$$

Which perfectly matches the *spherical* Bessel equation of degree 0. This is useful because the solution of this Bessel equation is known to be of the form:

$$A(x) = c_1 j_\alpha(x) + c_2 y_\alpha(x)$$

where j_α and y_α are *spherical* Bessel functions of the first and second kind respectively. Since we have an implicit boundary condition for T to remain finite and we previously set $\alpha = 0$ we know that:

$$A(x) = c_1 j_0(x)$$

because the $y_0(x)$ function is singular at $x = 0$ (so $c_2 = 0$). It turns out that in this special case we can write this explicitly as:

$$A(x) = c_1\frac{\sin(x)}{x}$$

Finally, we translate back into our r variable giving us the general solution of our ODE:

$$A(r) = c_1 j\left(\sqrt{\frac{\lambda}{D}}r\right) = c_1\frac{\sqrt{D}\sin\left(\sqrt{\frac{\lambda}{D}}r\right)}{\sqrt{\lambda}r} \propto \frac{\sin\left(\sqrt{\frac{\lambda}{D}}r\right)}{r}$$

Using our boundary condition at $r = R$ we must have:

$$A\left(\sqrt{\frac{\lambda}{D}}R\right) = 0$$

This tells us that $\sqrt{\frac{\lambda}{D}}R$ are roots of j_0. Since $j_0 = \frac{\sin(x)}{x}$, we know the roots are $z_n = n\pi$. Putting this together:

$$\sqrt{\frac{\lambda_n}{D}}R = n\pi \implies \lambda_n = \frac{n^2\pi^2 D}{R^2}$$

Therefore:

$$\sqrt{\frac{\lambda}{D}}r = \frac{n\pi r}{R}$$

So, we now have the general form of the solution to our PDE in u:

$$u(r,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} j_0\left(\frac{n\pi r}{R}\right)$$

Aside: Before moving on to the application of our initial condition in order to determine the coefficients C_n, it is worth considering the constant $n = 0$ term (i.e. the $\lambda_n = 0$ case) in our general solution by plugging $\lambda = 0$ into our radial/time-independent ODE (2.116), where we derive the following:

$$\begin{aligned}
r^2\frac{d^2 A}{dr^2} + 2r\frac{dA}{dr} = 0 \\
\implies \frac{d}{dr}\left(r^2\frac{dA}{dr}\right) = 0 \\
\implies r^2\frac{dA}{dr} = k \\
\implies \frac{dA}{dr} = \frac{k}{r^2},
\end{aligned}$$

and integrating gets us:

$$A(r) = -\frac{k}{r} + \kappa. \tag{2.117}$$

Imposing that the solution $A(r)$ is regular, i.e. if $A(0)$ exists, then we must have the first term in (2.117) vanishes, which requires that $k = 0$.

On the other hand, due to our auxiliary function method (2.114) we used to obtain homogeneous boundary conditions, (2.117) must also satisfy:

$$\begin{aligned} u(R,t) = A(R)B(t) = \kappa B(t) = 0, \\ \implies A(R) = 0. \end{aligned}$$

Therefore we have no constant terms present in the solution.

Returning to our general solution:

$$u(r,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} j_0\left(\frac{n\pi r}{R}\right).$$

Now we apply our initial conditions in order to find C_n, completing our solution:

$$u(r,0) = -96 = \sum_{n=1}^{\infty} C_n j_0\left(\frac{n\pi r}{R}\right).$$

Applying onto the basis of eigenfunctions $A_n(r) = j_0\left(\frac{n\pi r}{R}\right)$, recall that $w(r) = r^2$, and hence our orthogonality condition is given by:

$$\langle A_n(r), A_m(r)\rangle_{w(r)} = \int_0^R A_n(r)A_m(r)w(r)dr = 0, \text{ if } n \neq m.$$

Recall also that because $j_0(x) = \frac{sin(x)}{x}$, we recover the orthogonality condition of basis sine Fourier modes. Thus, projecting onto both sides, we arrive at:

$$\begin{aligned} \langle u(r,0), A_n(r)\rangle_{w(r)} &= \sum_{n=1}^{\infty} C_n \langle A_n(r), A_n(r)\rangle_{w(r)} \\ \implies (-96)\int_0^R j_0\left(\frac{n\pi r}{R}\right) r^2 dr &= C_n \int_0^R j_0\left(\frac{n\pi r}{R}\right) j_0\left(\frac{n\pi r}{R}\right) r^2 dr. \end{aligned}$$

Evaluating the projection integrals from both sides, we obtain:

$$\begin{aligned} &\implies \begin{cases} (-96)\int_0^R j_0\left(\frac{n\pi r}{R}\right) r^2 dr & = 96R^3 \frac{(-1)^n}{n^2\pi^2} \\ C_n \int_0^R j_0\left(\frac{n\pi r}{R}\right) j_0\left(\frac{n\pi r}{R}\right) r^2 dr & = C_n \frac{R^3}{2n^2\pi^2} \end{cases} \\ &\implies C_n = 96R^3 \frac{(-1)^n}{n^2\pi^2}\frac{2n^2\pi^2}{R^3} = 192(-1)^n \end{aligned}$$

Recalling that:

$$T(r,t) = 100 + u(r,t)$$

we have:

$$\begin{aligned} T(r,t) &= 100 + \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} j_0\left(\frac{n\pi r}{R}\right) \\ C_n &= 192(-1)^n \\ \lambda_n &= \frac{n^2\pi^2 D}{R^2} \end{aligned}$$

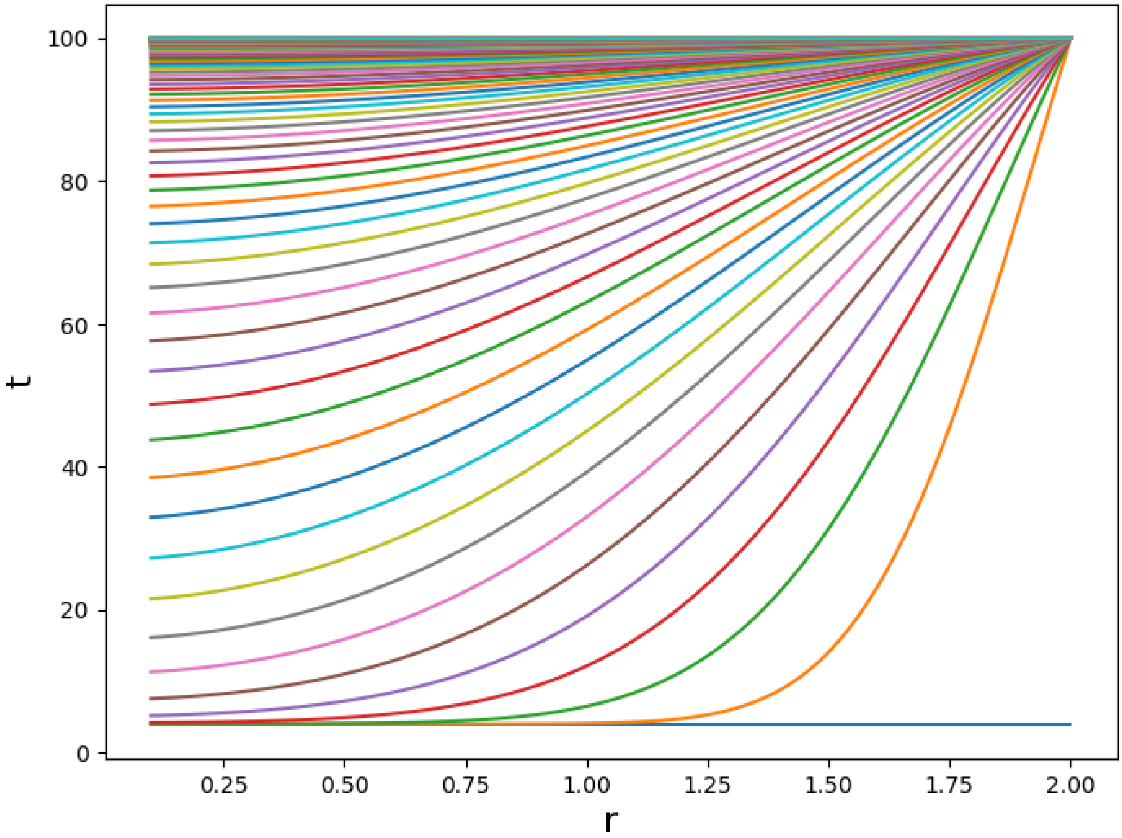

Figure 2.2: Heat distribution over the radius of the egg for increasing time. The Δt here is approximately 20 seconds.

We plot this solution in figure (2.2). By counting the lines representing 20 second intervals we find that the egg reaches 70 Celsius at around 400 seconds or ≈ 6.5 minutes which is about what we would expect.

Now considering our dimensional analysis, we calculated a diffusion timescale of approximately 33 minutes. We see that our diffusion timescale incorrectly calculates the actual time to cook an egg shown to be approximately 6 minutes. This can be attributed to the fact that dimensional analysis may be off in an order of magnitude. If we consider the true diffusion timescale in this case, we find that it is $\frac{1}{\pi^2}$ of our timescale calculated with dimensional analysis.

2.4.7 Radial vibrations of a circular drum

Lecture edited by Alyn, Arthur, Howard

Let's consider the motion of elastic waves on a circular drum of radius R. Let h be the height of the drum skin with respect to its rest position. The drum is assumed to be pinned at radius $r = R$, so $h(R, t) = 0$ at all times. Study this problem, and find axisymmetric solutions to the initial value problem with initial condition $h(r, 0) = 0$, $\partial h/\partial t = \exp(-10r^2/R^2)$ (which corresponds to hitting the drum in the center at $t = 0$, imparting it some velocity). You can assume that the wave speed c is constant.

Solution:
Since we are finding the axisymmetric solutions, we use the radially-symmetric wave equation with no angular dependencies:

$$\frac{\partial^2 h}{\partial t^2} = c^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right)\right]$$

Using separation of variables, we search for solutions of the form:

$$h(r, t) = A(r)B(t)$$

Applying this to the original equation and separating like terms:

$$\frac{1}{B(t)}\frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(r)}c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A(r)}{\partial r}\right)\right]$$

Notice that we have two independent functions of different forms and they are equal to one another. Thus they must both be equal to a constant, let the constant be $-\omega^2$:

$$\frac{1}{B(t)}\frac{d^2 B(t)}{dt^2} = \frac{1}{A(r)}c^2\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dA(r)}{dr}\right)\right] = -\omega^2$$

Note: $-\omega^2$ is chosen because we expect the final solution to be wave-like in time (e.g. a linear combination of sines and cosines)

We can now solve each individual ODE, starting with the time equation:

$$\frac{d^2 B(t)}{dt^2} = -\omega^2 B(t)$$

Recall the general solution of this form of ODE is $c_1 \cos(\omega t) + c_2 \sin(\omega t)$, thus:

$$B(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

Now we turn to the radial direction:

$$\frac{d}{dr}\left(r\frac{dA(r)}{dr}\right) = -\frac{\omega^2}{c^2}rA(r) = -\lambda r A(r)$$

Notice that this is a Sturm-Louiville equation. Recall the general form of such equations:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = -\lambda w(x)y$$

and notice that in our case $p(r) = r$, $q(r) = 0$, $w(r) = r$. And notice when $r = 0$, $p(0) = 0$. So the solution in the spatial direction is singular. However, because $p(r) = w(r)$, the nature of the singularity will have some regular behavior; so we call this "singular" with quotations.

Expanding the differential term on the LHS:

$$r\frac{d^2 A(r)}{dr^2} + \frac{dA(r)}{dr} = -\lambda r A(r)$$

Rearranging terms yields the familiar form:

$$r\frac{d^2 A(r)}{dr^2} + \frac{dA(r)}{dr} + \lambda r A(r) = 0$$

of a Bessel Equation!

Recall the general form of a Bessel Equation is $x^2y'' + y' + (x^2 - \alpha^2)y = 0$.

Let us perform the following change of variable: $x = r\sqrt{\lambda}$ in order to reach this exact form. Then:

$$x^2\frac{d^2A(x)}{dx^2} + x\frac{dA(x)}{dx} + x^2A(x) = 0$$

Notice that $\alpha = 0$. The general solution for Bessel Equations of this form are a linear combination of Bessel functions of order O:

$$A(x) = c_1J_0(x) + c_2Y_0(x)$$

Changing back to our radial coordinate variable we get the general solution in the spatial dimension as:

$$A(r) = c_1J_0(r\sqrt{\lambda}) + c_2Y_0(r\sqrt{\lambda})$$

Now we can apply the boundary conditions in the radial direction to solve for the coefficients:

$$h(0,t);\ A(0) = \text{finite} = c_1J_0(0) + c_2Y_0(0)$$

Recall that $Y_0(0) = \infty$. Thus for the condition that $h(0,t) = $ finite, it must be the case that $c_2 = 0$. Applying $h(R,t) = 0$:

$$h(R,t);\ A(R) = 0 = c_1J_0(R\sqrt{\lambda})$$

In order for this condition to be satisfied, we must enforce that $R\sqrt{\lambda} = z_n$ where z_n represent the zeros of the Bessel Function J_0. Note that these are well-known and can be found in textbooks (cf. Abramowitz and Stegun) or in Wolfram Alpha or Matlab. Using this condition, we can solve for λ:

$$\lambda_n = \frac{z_n^2}{R^2} = \frac{\omega_n^2}{c^2}$$

This gives an oscillation frequency:

$$\omega_n = \frac{cz_n}{R}$$

Thus, the final solution that satisfies the boundary conditions in the radial direction is

$$A(r) = c_1J_0(r\sqrt{\lambda_n}) = c_1J_0\left(r\frac{z_n}{R}\right)$$

Reconstructing $h(r,t)$:

$$h(r,t) = A(r)B(t) = \sum_{n=1}^{\infty} J_0\left(r\frac{z_n}{R}\right)[a_n\cos(\omega_nt) + b_n\sin(\omega_nt)]$$

To satisfy the initial condition $h(r,0) = 0$, we simply need $a_n = 0$ for all n. Now we work on satisfying the initial condition $\frac{\partial h(r,0)}{\partial t} = \exp(-10r^2/R)$.

$$\frac{\partial h(r,t)}{\partial t} = \sum_{n=1}^{\infty} b_nJ_0\left(r\frac{z_n}{R}\right)\omega_n\cos(\omega_nt)$$

$$\frac{\partial h(r,0)}{\partial t} = \sum_{n=1}^{\infty} b_n\omega_nJ_0\left(r\frac{z_n}{R}\right) = \exp(-10r^2/R)$$

We now project onto the J_0 basis. Note: the weight function is $w(r) = r$

$$\int_0^R \sum_{n=1}^{\infty} b_n\omega_nJ_0\left(r\frac{z_n}{R}\right)J_0\left(r\frac{z_m}{R}\right)rdr = \int_0^R \exp(-10r^2/R)J_0\left(r\frac{z_m}{R}\right)rdr$$

The integral on the left-hand side is equal to 0 unless $m = n$, so

$$\omega_mb_m\int_0^R J_0^2\left(r\frac{z_m}{R}\right)rdr = \int_0^R \exp(-10r^2/R)J_0\left(r\frac{z_m}{R}\right)rdr$$

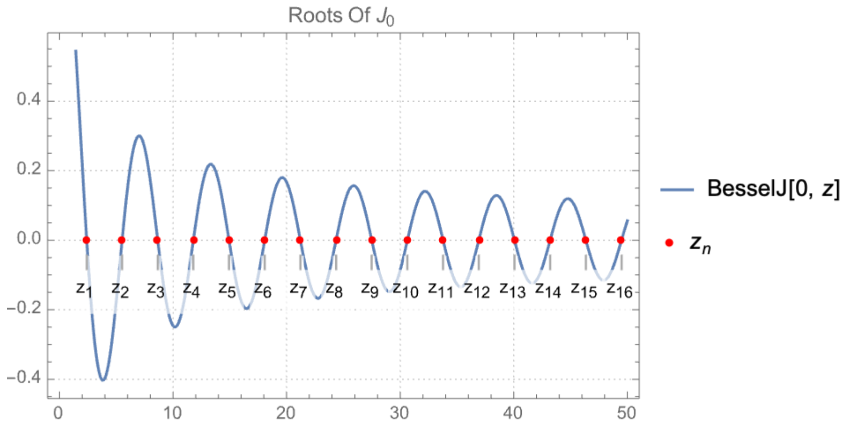

Figure 2.3: Roots of the First-Order Bessel Function of the First Kind

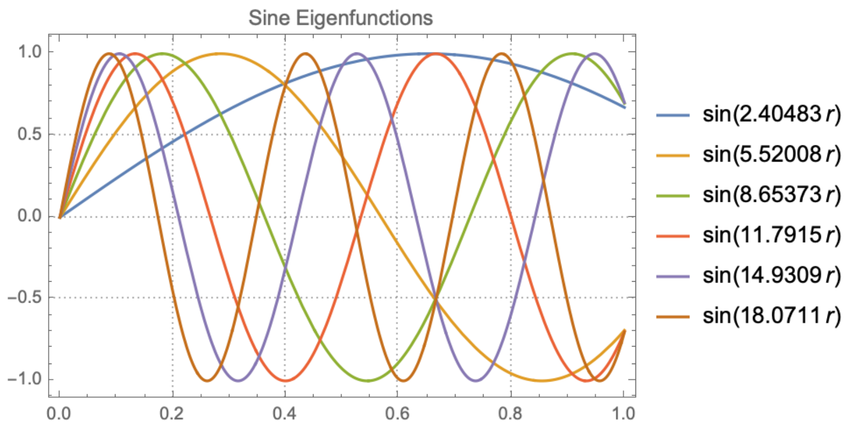

Figure 2.4: Eigenfunctions of Sines

$$b_m = \frac{\int_0^R \exp\left(-10r^2/R\right) J_0\left(r\frac{z_m}{R}\right) r dr}{\omega_m \int_0^R J_0^2\left(r\frac{z_m}{R}\right) r dr}$$

The final solution is:

$$h(r,t) = \sum_{n=1}^{\infty} b_n J_0\left(r\frac{z_n}{R}\right)\sin(\omega_n t)$$

where z_n represent the zeros of the Bessel Function J_0,

$$\omega_n = \frac{cz_n}{R}$$

Recall that c was assumed to be a constant wave speed.

$$b_n = \frac{\int_0^R \exp\left(-10r^2/R\right) J_0\left(r\frac{z_n}{R}\right) r dr}{\omega_n \int_0^R J_0^2\left(r\frac{z_n}{R}\right) r dr}$$

The following figures illustrate the behavior of the drum at various times for $c = 1$, $R = 1$.

2.4.8 Other interesting properties of Sturm-Liouville problems

There are a few other interesting properties of Sturm-Liouville problems that are worth mentioning because they can be used to obtain approximate estimate for the eigenvalues, and sometimes that is all that is needed.

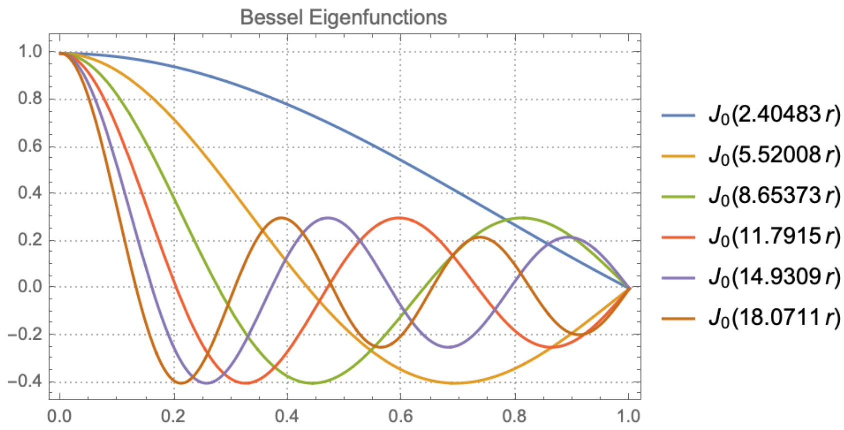

Figure 2.5: Eigenfunctions of J_0

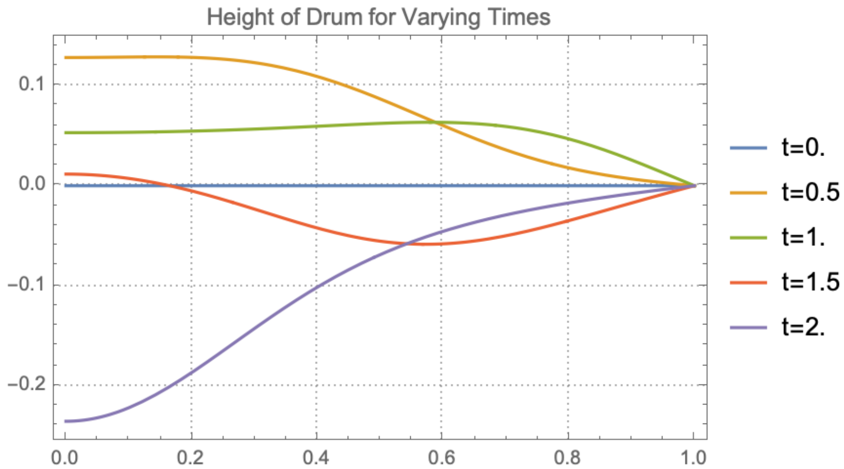

Figure 2.6: Solution of the Wave Equation on a Drum for Various Times

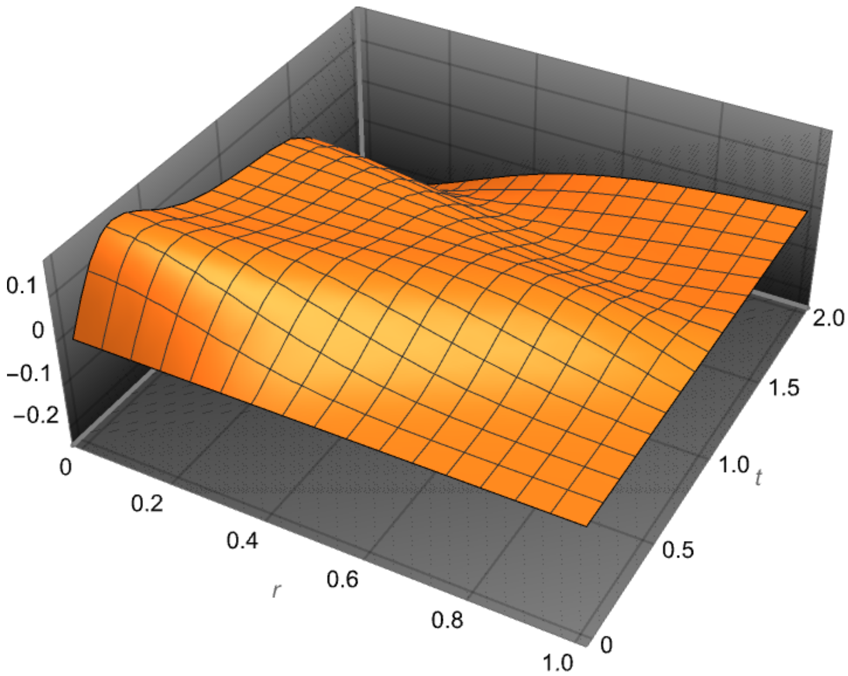

Figure 2.7: Solution of the Wave Equation on a Drum for Continuous Time

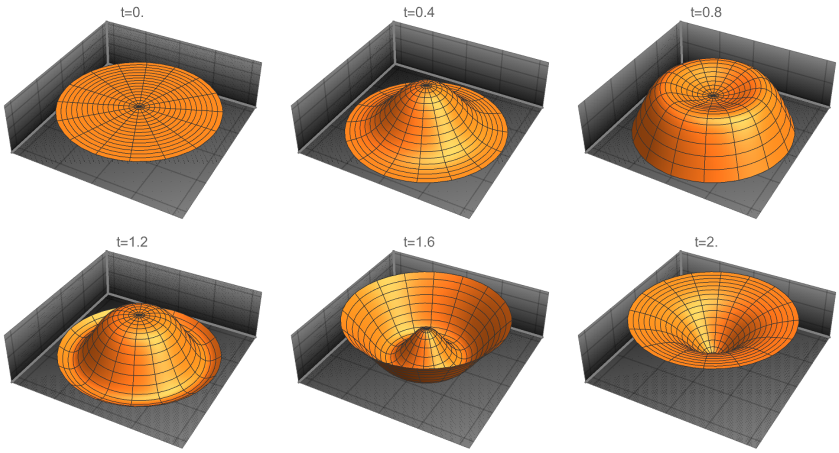

Figure 2.8: Solution of the Wave Equation on a Drum for Various Times

Approximating eigenvalues using the Rayleigh Quotient
The Rayleigh Quotient associated with a Sturm-Liouville problem is a *functional* (ie., a function of a function), defined as

$$\mathcal{R}(u) = -\frac{\int_{x_a}^{x_b} u(x)\mathcal{L}(u(x))dx}{\int_{x_a}^{x_b} u^2(x)w(x)dx} \tag{2.118}$$

Note that for any function u, $\mathcal{R}(u)$ returns a scalar.

The key is that the Rayleigh Quotient is a functional that has very nice properties:

- For a regular SL problem, the absolute minimum of $\mathcal{R}(u)$ over all possible functions u that are continuous in (a, b) and satisfy the boundary conditions is the principal eigenvalue λ_0, and the minimum is achieved when $u(x)$ is proportional to $v_0(x)$ (i.e. for the eigenfunction corresponding to λ_0).
- Each successive eigenvalue λ_n is a *local* minimum of $\mathcal{R}(u)$, achieved when $u(x) = v_n(x)$ (i.e. for the eigenfunction corresponding to λ_n

Proof of first bullet point: The proof is really quite simple. First, note that if the Sturm-Liouville problem is regular we can expand u in terms of the eigenfunctions $v_n(x)$ of $\mathcal{L}$:

$$u(x) = \sum_{n=0}^{\infty} a_n v_n(x) \tag{2.119}$$

where by definition

$$\mathcal{L}v_n = -\lambda_n w(x) v_n(x) \tag{2.120}$$

The numerator of $\mathcal{R}(u)$ becomes

$$\int_{x_a}^{x_b} u(x)\mathcal{L}(u(x))dx = \int_{x_a}^{x_b} \sum_{n=0}^{\infty} a_n v_n(x) \sum_{m=0}^{\infty} a_m \mathcal{L}(v_m)dx \tag{2.121}$$

$$= -\int_{x_a}^{x_b} \sum_{n=0}^{\infty} a_n v_n(x) \sum_{m=0}^{\infty} a_m \lambda_m v_m(x) w(x) dx = -\sum_{n=0}^{\infty} \lambda_n a_n^2 \langle v_n, v_n\rangle$$

while the denominator simply becomes (using similar steps)

$$\int_{x_a}^{x_b} u^2(x)w(x)dx = \sum_{n=0}^{\infty} a_n^2 \langle v_n, v_n\rangle \tag{2.122}$$

so

$$\mathcal{R}(u) = \frac{\sum_{n=0}^{\infty} \lambda_n a_n^2 \langle v_n, v_n\rangle}{\sum_{n=0}^{\infty} a_n^2 \langle v_n, v_n\rangle} \tag{2.123}$$

Since $\lambda_n > \lambda_0$ for all $n > 1$, we have

$$\mathcal{R}(u) \geq \frac{\sum_{n=0}^{\infty} \lambda_0 a_n^2 \langle v_n, v_n \rangle}{\sum_{n=0}^{\infty} a_n^2 \langle v_n, v_n \rangle} = \lambda_0 \tag{2.124}$$

Furthermore, because $a_n^2 \langle v_n, v_n \rangle \geq 0$, the only way to achieve the minimum value $\mathcal{R}(u) = \lambda_0$ is to make sure that none of the eigenfunctions with $n > 0$ participate in the solution (otherwise their contributions will raise $\mathcal{R}(u)$). This forces the minimum to happen when $u(x) = a_0 v_0(x)$.

Beyond this result, it can also be shown that if $u(x)$ approximates $v_n(x)$ relatively well then $\mathcal{R}(u)$ approximates λ_n even better. (This will be shown formally in the context of your Numerical Linear Algebra course next quarter, and is the mathematical basis for some important iterative methods to find eigenvalues and eigenvectors iteratively.) As a result, by picking a 'test' function u that has the expected behavior of v_n, we can get quite good estimates for the eigenvalue λ_n by calculating $\mathcal{R}(u)$.

Example: Consider the SL problem:

$$\frac{d^2u}{dx^2} + (\lambda - x^2)u = 0 \tag{2.125}$$

$$u'(0) = 0, u(1) = 0 \tag{2.126}$$

Find an approximation to λ_0 (and compare it with the exact solution, which has $\lambda_0 \simeq 2.597...$).

Solution: In this problem $\mathcal{L}(u) = \frac{d^2u}{dx^2} - x^2 u$, and $w(x) = 1$. Let's pick a function u that satisfies the boundary conditions: say $u(x) = 1 - x^2$ for instance. Then $\mathcal{L}(u) = -2 - x^2(1 - x^2)$ and

$$\mathcal{R}(u) = -\frac{\int_0^1 (1-x^2)(-2 - x^2(1-x^2))dx}{\int_0^1 (1-x^2)^2 dx} = -\frac{-148/105}{8/15} = \frac{37}{14} = 2.642 \tag{2.127}$$

which is fairly close to the true eigenvalue.

Note that we can leverage what we know of the zeroes of eigenfunctions of regular SL problems to guess what they 'look like':

- if we had want to approximate λ_1, we would pick a test function which as 1 zero in the interval;
- if we want to approximate λ_n, we would pick a test function which as n zeros in the interval.

Approximating large eigenvalues (and eigenfunctions):
As we have seen in the last lecture, eigenvalues of regular SL problems form a strictly monotonic sequence of numbers that is unbounded (i.e. tends to ∞). As it turns out, an important method of asymptotic analysis called WKB theory can be used to approximate the eigenfunctions and eigenvalues of a regular SL problem for large values of n. We will show later in this course that for large n,

$$v_n(x) \simeq [w(x)p(x)]^{-1/4} \left[\alpha \cos\left(\sqrt{\lambda_n} \int_{x_a}^{x} \sqrt{\frac{w(x')}{p(x')}} dx' \right) + \beta \sin\left(\sqrt{\lambda_n} \int_{x_a}^{x} \sqrt{\frac{w(x')}{p(x')}} dx' \right) \right] \tag{2.128}$$

Applying boundary conditions yields λ_n. For example, if we want to have homogeneous Dirichlet conditions, such that $v_n(x_a) = v_n(x_b) = 0$, we need the sine function only ($\alpha = 0$), and furthermore require that

$$\sqrt{\lambda_n} \int_{x_a}^{x_b} \sqrt{\frac{w(x')}{p(x')}} dx' = n\pi \tag{2.129}$$

This implies

$$\lambda_n = \frac{n^2 \pi^2}{\left(\int_{x_a}^{x_b} \sqrt{\frac{w(x')}{p(x')}} dx' \right)^2} \tag{2.130}$$

Example of application: Waves in the Sun

Sound waves propagate in the stars with sound speed c which is a function of radius r only. We assume that spherically-symmetric sound waves, which are just radial pulsations of the star, follow the equation

$$\frac{\partial^2 f}{\partial t^2} = \frac{c^2(r)}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) \tag{2.131}$$

for $r \in (0, R)$ where R is the star's radius. We also assume $f(R, t) = 0$ as a boundary condition (neither of these statements is completely true, but that's ok for now).

After separating the variables as $f(r, t) = A(r)B(t)$, we get

$$\frac{d^2 B}{dt^2} = -\omega^2 B \tag{2.132}$$

$$\frac{d}{dr}\left(r^2\frac{dA}{dr}\right) = -\frac{\omega^2 r^2}{c^2(r)}A \tag{2.133}$$

where, as we expect oscillations, we have immediately written the common constant as $-\omega^2$.

The second equation is a singular Sturm-Liouville problem, with $p(r) = r^2$, $q(r) = 0$, $w(r) = r^2/c^2(r)$ and $\lambda = \omega^2$. In general, finding the eigenvalues λ requires solving the problem numerically. However, for *large* eigenvalues, we can use the WKB solution (even though the problem is singular, it behaves as a regular problem).

The solutions for A are simply:

$$A_n(r) \simeq \frac{c^{-1/2}(r)}{r}\left[\alpha\cos\left(\sqrt{\lambda_n}\int_0^r c^{-1}(r')dr'\right) + \beta\sin\left(\sqrt{\lambda_n}\int_0^r c^{-1}(r')dr'\right)\right] \tag{2.134}$$

Clearly, if we want to avoid the solution blowing up at $r = 0$ we will need to get rid of the cos part of the solution, setting $\alpha = 0$. Then to have $A_n(R) = 0$ as well, we need

$$\lambda_n = \frac{n^2\pi^2}{\left(\int_0^R c^{-1}(r')dr'\right)^2} \to \omega_n = \frac{n\pi}{\left(\int_0^R c^{-1}(r')dr'\right)} \tag{2.135}$$

Now how the denominator in the expression for ω_n is simply the square of the sound travel time between the center of the Sun and the surface, so by measuring the oscillation frequencies of the Sun, we can figure out what that travel time is.

2.5 Second order 3D or 4D linear PDEs

In what we have learned so far, we studied PDEs that had only 2 independent variables, either one spatial and one time variables, or two spatial variables. We now look at more complicated problems with more variables. As we shall see, it is still possible, for PDEs of the right form, to apply exactly the same principles as we have done so far, although the book-keeping needed rapidly becomes quite complicated as more variables are added.

2.5.1 Forced vibrations of a square plate

Let's consider forced vibrations of a square plate of size length L, described by the equation

$$\frac{\partial^2 h}{\partial t^2} = c^2 \left[\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right] + F(x,y)\sin(\omega t) \tag{2.136}$$

where $F(x,y)$ is some arbitrary shape function. We assume that $h = 0$ on the edge of the plate, and that at $t = 0$ the plate is at rest. Let's find solutions of this problem.

As usual for forced problems, we first look at the homogeneous problem (without forcing), and seek separable solutions. Since the boundary conditions are homogeneous, we can do that easily. Let's look for solutions of the form

$$h(x,y,t) = T(t)X(x)Y(y)$$

Then,

$$\frac{1}{T}\frac{d^2T}{dt^2} = c^2 \left[\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dx^2} \right] = -\omega^2 \tag{2.137}$$

Both sides indeed have to be constant (since the left-hand side is a function of time only, and the right-hand side is a function of space only) and the constant has to be negative since we expect oscillations. The spatial problem becomes

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dx^2} - \frac{\omega^2}{c^2} = -k_x^2$$

where again we have separated the two sides, and set each to be a constant. We cannot fit the solution $X(x)$ to the boundary conditions if that constant is positive, hence the choice to write it as $-k_x^2$. The solutions are (as usual)

$$X(x) = \sin(k_x x) = \sin\left(\frac{n\pi x}{L}\right)$$

with k_x taking all the possible values $k_{xn} = n\pi/L$. Finally,

$$\frac{1}{Y}\frac{d^2Y}{dx^2} = -\frac{\omega^2}{c^2} + \frac{n^2\pi^2}{L^2}$$

and similarly we find

$$Y(y) = \sin(k_y y) = \sin\left(\frac{m\pi y}{L}\right)$$

with k_y taking all the possible values $k_{ym} = m\pi/L$.

The product of $X(x)$ and $Y(y)$ is the two-dimensional eigenfunction of the spatial operator:

$$S_{nm}(x,y) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)$$

as it is easy to check that

$$\frac{\partial^2 S_{nm}}{\partial x^2} + \frac{\partial^2 S_{nm}}{\partial y^2} = -\lambda_{nm} S_{nm} \text{ where } \lambda_{nm} = k_{xn}^2 + k_{ym}^2 = (n^2 + m^2)\frac{\pi^2}{L^2} \tag{2.138}$$

Going back to equation (2.137) shows that each of these spatial eigenmodes would oscillate independently with frequency

$$\omega_{nm} = c\sqrt{\lambda_{nm}} \tag{2.139}$$

in the homogeneous case.

However, here we have to deal with the forced case. We therefore let the complete solution be

$$h(x,y,t) = \sum_{n,m} S_{nm}(x,y)T_{nm}(t) \tag{2.140}$$

Substituting this into the governing equation reveals that

$$\sum_{n,m} \frac{d^2 T_{nm}}{dt^2} S_{nm}(x,y) = -\sum_{n,m} \lambda_{nm} S_{nm}(x,y) T_{nm}(t) + F(x,y)\sin(\omega t) \quad (2.141)$$

We can use the orgthogonality of sines in each direction to note that the 2D eigenmodes are orthogonal to each other as well, satisfying

$$\int_0^L \int_0^L S_{nm}(x,y) S_{n'm'}(x,y) dx dy = \int_0^L \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n'\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \sin\left(\frac{m'\pi y}{L}\right) dx dy$$
$$= \frac{L^2}{4} \delta_{nn'} \delta_{mm'} \quad (2.142)$$

Projecting the equation onto the 2D eigenmodes, we then get

$$\frac{d^2 T_{nm}}{dt^2} = -\omega_{nm}^2 T_{nm}(t) + F_{nm} \sin(\omega t)$$

where ω_{nm} was defined earlier, and

$$F_{nm} = \frac{4}{L^2} \int_0^L \int_0^L S_{nm}(x,y) F(x,y) dx dy$$

The solution to this equation is of the form

$$T_{nm}(t) = a_{nm} \cos(\omega_{nm} t) + b_{nm} \sin(\omega_{nm} t) + K_{nm} \sin(\omega t)$$

where a_{nm} and b_{nm} are integration constants that will ultimately be fitted to the initial conditions, and K_{nm} is found to ensure that $K_{nm} \sin(\omega t)$ is a particular solution of the forced problem, namely

$$-\omega^2 K_{nm} = -\omega_{nm}^2 K_{nm} + F_{nm} \to K_{nm} = \frac{F_{nm}}{\omega_{nm}^2 - \omega^2}$$

The complete solution is therefore

$$h(x,y,t) = \sum_{n,m} \left[a_{nm} \cos(\omega_{nm} t) + b_{nm} \sin(\omega_{nm} t) + K_{nm} \sin(\omega t)\right] S_{nm}(x,y) \quad (2.143)$$

Finally, we can apply the initial conditions. If the plate is completely at rest at $t = 0$, then

$$h(x,y,0) = \frac{\partial h}{\partial t}(x,y,0) = 0 \quad (2.144)$$

The first condition requires $a_{nm} = 0$ for all n, m. The second condition requires

$$b_{nm}\omega_{nm} + K_{nm}\omega = 0 \to b_{nm} = -K_{nm} \frac{\omega}{\omega_{nm}}$$

which completes the problem. This solution is implemented in the Matlab script provided.

Note: As in the case of the 1D vibrating string we see that:

- There are a number of spatial eigenfunctions of the problem, that depend only on the spatial operator and the boundary conditions
- To each of these eigenmodes corresponds a given oscillation frequency.
- The fundamental frequency corresponds to the 'simplest' spatial eigenfunction, and higher frequencies correspond to more complex ones (with more zeros)

What is new with more dimensions is that

- Each eigenfunction is now characterized by two indices m, n, corresponding to the two spatial variables.
- As long as each of the eigenvalue problems in the spatial variables is a Sturm-Liouville problem, the orthogonality of the eigenfunctions in that variable is guaranteed. Then, the 2D eigenfunction of the full spatial problem (which is the product of the single-variable eigenfunctions) are also orthogonal to one another.
- Eigenvalues can now be degenerate, in some cases. In this particular example, we have $\omega_{mn} = \omega_{nm}$, even though these correspond to different spatial eigenmodes.

2.5.2 Cooling of a rocky planet

A rocky planet was formed by the collision of many planetesimals. The collision process results in the heating of the interior of the planet to very high temperatures, but also to spatial variations of that temperature. After assembly, the rocky planet then slowly cools off. In this problem, we are going to ignore convective heat transport processes, and simply model the evolution of the temperature profile in the planet with a diffusion equation

$$\frac{\partial T}{\partial t} = D\nabla^2 T \tag{2.145}$$

where the boundary condition is $T(R, \theta, \phi, t) = 0$ at the surface of the planet $r = R$. The initial condition is $T(r, \theta, \phi, 0) = T_0(r, \theta, \phi)$.

- Write the problem in spherical coordinates (r, θ, ϕ)
- Successively separate the variables: first t, then ϕ, then θ and finally r.
- Find the eigenfunctions and their eigenvalues in each spatial direction. You will probably need to consult the Abramowitz and Stegun book!
- Write down the orthogonality relation of the eigenfunctions corresponding to each spatial coordinate.
- Use that to fit the initial conditions.

Solution: was obtained in Section.

Note: Solving this problem has introduced the 2D eigenfunctions of the Laplacian operator on a spherical shell, which are called *spherical harmonics*. They are usually written as

$$Y_l^m(\theta, \phi) = e^{\pm im\phi} P_l^m(\sin\theta) \tag{2.146}$$

where P_l^m is a Legendre function of order m and degree l, and where $e^{\pm im\phi}$ denotes a linear combination of these complex exponentials, or equivalently, linear combinations of $\sin(m\phi)$ and $\cos(m\phi)$.

Spherical harmonics come up very frequently in many problems that take place on a *full* spherical shell or in the *full* sphere (e.g. atomic physics, electrostatics, planetary and stellar astrophysics, etc.). As a result, they have been studied extensively. They have the following properties:

- If ∇_h^2 is the 2D Laplacian in spherical coordinates that only involves the 'horizontal' coordinates θ, ϕ, then
$$\nabla_h^2 Y_l^m(\theta, \phi) = -\frac{l(l+1)}{r^2} Y_l^m(\theta, \phi) \tag{2.147}$$
- The $Y_l^m(\theta, \phi)$ are only defined for $|m| \le l$.
- They form a complete basis for all functions on a spherical shell, so that
$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (a_{lm}\cos(m\phi) + b_{lm}\sin(m\phi) P_l^m(\sin\theta) \tag{2.148}$$
- They are mutually orthogonal, and satisfy the orthogonality relationship: to be completed

2.6 Green's function solutions of forced time-independent problems

Lecture edited by Charlie and Sean.

In previous lectures, we have come across a few examples of forced (non-homogeneous) second-order linear PDEs. Let us now study this more generally, and introduce the concept of Green's functions. In this lecture we begin with the (slightly easier) case of ODE problems, and in the next lecture we will continue with time-dependent PDEs.

The concept of Green's functions relies on the δ function, whose properties are summarized in the next section.

2.6.1 The Dirac delta function

The Dirac delta function δ is a function that has the following two defining properties:

$$\delta(x) = 0 \quad \forall x \neq 0 \tag{2.149}$$

$$\int f(x)\delta(x-a)dx = f(a) \quad \forall a \tag{2.150}$$

for any function $f(x)$ continuous in the vicinity of a. It does not matter what the bounds of the integral are as long as they contain $x = a$. Substituting $f(x) = 1$ as a possible function, we immediately get the first property of the Dirac δ that

$$\int \delta(x)dx = 1 \tag{2.151}$$

One practical way of defining this function, which also has the advantage of providing a 'visual' of what it may look like, is

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{2.152}$$

that is, an infinitely narrow, infinitely tall Gaussian function. It is easy to verify that the normalization condition (2.151) is satisfied. This is not the only possible way of defining δ, however, and other limits of similar families of functions also work.

The concept of a δ function can easily be generalized to multiple dimensions if needed. Let $\mathbf{r}$ be a position vector. Then, the multivariate δ function has the properties that

$$\delta(\mathbf{r}) = 0 \quad \forall |\mathbf{r}| \neq 0 \tag{2.153}$$

$$\int f(\mathbf{r})\delta(\mathbf{r} - \mathbf{r_0})d^n\mathbf{r} = f(\mathbf{r_0}) \tag{2.154}$$

$$\int \delta(\mathbf{r} - \mathbf{r_0})d^n\mathbf{r} = 1 \tag{2.155}$$

where this time the integral is taken over a 'ball' (or disk in 2D) centered around $\mathbf{r_0}$, and $d^n\mathbf{r}$ denotes an area or volume integral in n dimensions.

2.6.2 Green's functions for ODEs

Let's begin by considering ODEs of the form

$$\mathcal{L}_x(f) = F(x) \tag{2.156}$$

where $\mathcal{L}_x$ is a second-order ordinary linear operator in the single variable x only, and F is some known function of x only. In addition, we assume that this ODE is subject to some homogeneous boundary conditions (either Dirichlet, Von Neumann, or Robin conditions). If the boundary conditions are not homogenous, we use the usual trick to recast the problem into one with homogeneous conditions.

We know from the theory of ODEs that the solution to this problem can be written as the sum of the general solution of the homogeneous equation $\mathcal{L}_x(f) = 0$, plus a particular solution of the forced equation. The question then shifts to how to find that particular solution. In specific instances, where $F(x)$ is a polynomial function, or an exponential function, or a sine or cosine function, there are standard solutions or tricks that can be used – and we have done so already earlier in this course. However, when $F(x)$ is not of that standard form, it can be more difficult to find the particular solution with naive guesses. Instead, we need to use a more systematic approach, which involves the notion of Green's functions. Let's first introduce the tools needed to do that.

The Green's function associated with the operator $\mathcal{L}_x$ and some given homogeneous boundary conditions of the problem is the solution of the equation

$$\mathcal{L}_x(G) = \delta(x - x') \tag{2.157}$$

subject to these boundary conditions, namely, the response to forcing by a δ-function centered at $x = x'$ (so-called 'impulse' forcing). The Green's function depends both on x and on the position x' at which the impulse is applied, so in all generality $G = G(x, x')$.

Crucially, if we know what $G(x, x')$ is, we can find solutions to any forced problem using linearity of $\mathcal{L}_x$. Indeed, let's construct the function

$$f(x) = \int G(x, x')F(x')dx' \tag{2.158}$$

In this expression the integral bounds cover the range of the variable x. If this is a boundary value problem, then the integral covers the interval from one boundary to the other. If x is unbounded, then one or both bounds of the integral are $\pm\infty$.

We see that

$$\mathcal{L}_x(f) = \int \mathcal{L}_x(G(x, x'))F(x')dx' = \int \delta(x - x')F(x')dx' = F(x) \tag{2.159}$$

is indeed a solution of the forced problem $\mathcal{L}_x(f) = F(x)$. Note that we can move $\mathcal{L}_x$ under the integral sign because L_x acts on the x variable, while the integral is in x'.

The question then shifts to how to find these Green's function solutions.

2.6.3 Green's function solutions in 2-point boundary value ODEs

There are (at least) two practical ways of finding the Green's function for 2-point boundary value ODE problems: expansion onto the eigenfunctions of L_x, and patching solutions of the homogeneous problem at the impulse point. The first is very similar to what we have seen before for PDEs and follows the same steps, as we now see.

Example 1: Consider the ODE problem

$$\mathcal{L}_x(f) = \frac{d^2 f}{dx^2} + \frac{df}{dx} = F(x) \tag{2.160}$$

with $f(0) = f(L) = 0$.

- Find the eigensolutions of $\mathcal{L}_x(f) = -\lambda f(x)\sigma(x)$, where you should choose $\sigma(x)$ conveniently to turn the problem into an easily solvable one. Note that σ is not necessarily the weight function of the SL problem (this depends on whether $\mathcal{L}_x$ is already in SL form or not).
- Expand f onto the basis of eigenfunctions, and project the forced equation onto that basis. Note that you will have to identify the associated SL problem to do that.
- Solve each resulting algebraic equation, and assemble them to create the solution to the forced problem.
- Show that the solution of the forced problem can be written as $f(x) = \int_0^L G(x, x')F(x')dx'$. What is $G(x, x')$?

Solution: We want to find the solutions for

$$\frac{d^2 f}{dx^2} + \frac{df}{dx} = -\lambda f(x)\sigma(x)$$

where we will set $\sigma(x) = 1$ to make the problem easy to solve. This gives us the following characteristic polynomial:

$$r^2 + r + \lambda = 0$$

which has roots given by

$$r = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$$

The boundary conditions imply that the solution should be oscillatory rather than exponential, which gives the condition that $1 - 4\lambda < 0$ or $\lambda > \frac{1}{4}$.

We have solutions of the form

$$f(x) = Ae^{\frac{-1+i\sqrt{4\lambda-1}}{2}x} + Be^{\frac{-1-i\sqrt{4\lambda-1}}{2}x} = Ae^{-\frac{1}{2}x}e^{\frac{i\sqrt{4\lambda-1}}{2}} + Be^{-\frac{1}{2}x}e^{\frac{-i\sqrt{4\lambda-1}}{2}}$$
$$\implies f(x) = \alpha e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{4\lambda-1}}{2}x\right) + \beta e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{4\lambda-1}}{2}x\right)$$

Implementing boundary conditions $f(0) = f(L) = 0$, we get $\alpha = 0$ and $\lambda_n = \frac{1}{4} + \left(\frac{n\pi}{L}\right)^2$.

The corresponding eigenfunctions are therefore

$$f_n(x) = e^{-\frac{1}{2}x}\sin\left(\frac{n\pi x}{L}\right)$$

We know from SL theory that these eigenfunctions must be orthogonal but we need to find the appropriate weight function. Simply by inspection of $f_n(x)$, and using the orthogonality of sines, we can guess that the weight function is e^x (this can also be verified by putting the original equation into SL form), so that the correct inner product is

$$\langle f_n, f_m\rangle = \int_0^L f_n(x)f_m(x)e^x dx = \frac{L}{2}\delta_{nm}$$

Now that we know the eigenfunctions and eigenvalues to the unforced ODE are, we can begin finding the Green's function. We can say that the solution to the forced problem must be of the form

$$f(x) = \sum_{n=1}^{\infty} a_n f_n(x)$$

since the $\{f_n\}$ form a complete basis for all functions on $[0, L]$. Substituting this into (2.160) we get

$$\sum_{n=1}^{\infty} a_n\left(\frac{d^2 f_n}{dx^2} + \frac{df_n}{dx}\right) = \sum_{n=1}^{\infty} a_n(-\lambda_n f_n) = F(x)$$

projecting this onto $f_m(x)$ as defined above we obtain

$$-a_m\lambda_m\frac{L}{2} = \int_0^L F(x)e^x e^{-\frac{1}{2}x}\sin\left(\frac{m\pi x}{L}\right)dx$$

Therefore we can solve for a_m and find that

$$a_m = -\frac{2}{L\lambda_m}\int_0^L F(x')e^{\frac{x'}{2}}\sin\left(\frac{m\pi x'}{L}\right)dx'$$

where we have re-written the integral in terms of the dummy variable x' to avoid confusion later. The overall solution therefore is

$$f(x) = \sum_{m=1}^{\infty} a_m e^{-\frac{1}{2}x}\sin\left(\frac{m\pi x}{L}\right)$$

with a_m given above. Expanding this equation we can determine the Green's function by identifying

$$\sum_{n=1}^{\infty}\left[-\frac{2}{L\lambda_n}\int_0^L F(x')e^{\frac{x'}{2}}\sin\left(\frac{n\pi x'}{L}\right)dx'\right]e^{-\frac{1}{2}x}\sin\left(\frac{n\pi x}{L}\right) \equiv \int_0^L G(x', x)F(x')dx'$$

which reveals that

$$G(x, x') = \sum_{n=1}^{\infty} -\frac{2}{L\lambda_n}e^{-\frac{x-x'}{2}}\sin\left(\frac{n\pi x'}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

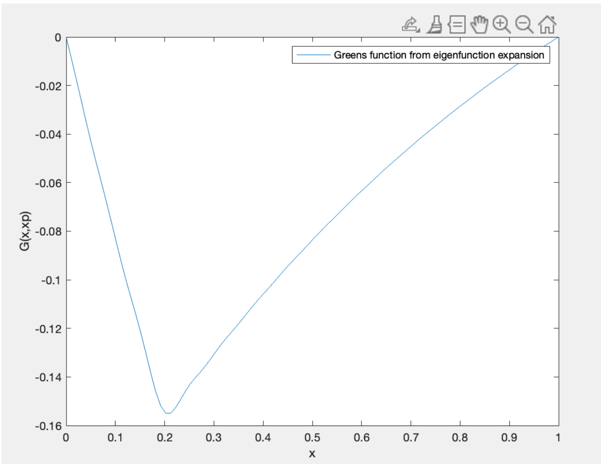

Figure 2.9: This shows the plotted greens function for $x' = 0.2$

Lecture edited by Yiqin, Janice and Julian

The second method is a little different, and leverages the fact that $\delta(x - x')$ is zero everywhere other than at $x = x'$. Therefore, on each sub-interval (x_a, x'), (x', x_b) the equation for the Green's function is homogeneous. The idea is to solve it in each of these subintervals, and require continuity of G (and some of its higher-order derivatives, if $\mathcal{L}_x$ is greater than second order, see RHB 15.2.5) at $x = x'$. In addition, by integrating the equation across $x = x'$ we can obtain an additional condition on the derivative of f at $x = x'$. Let's see how this work through an example.

Example 2: Consider the equation

$$\frac{d^2G}{dx^2} + \frac{dG}{dx} = \delta(x - x') \tag{2.161}$$

with $G(0, x') = G(L, x') = 0$ (this simply defines the Green's function for Example 1 above). This time we will solve for G directly without projecting onto eigenfunctions.

- Find the respective solutions to the homogeneous equation on $(0, x')$, (x', L), applying the relevant boundary condition for each sub-interval.
- What condition does the requirement of continuity of $G(x, x')$ across $x = x'$ imply? Integrate the equation from $x = x' - \epsilon$ to $x = x' + \epsilon$ and take the limit of very small ϵ. What additional condition on the derivative of G across $x = x'$ do you obtain?
- Use these two conditions to form $G(x, x')$
- Plot $G(x, x')$ for a few different values of x' and compare it to the solution obtained in Example 1.

Solution:

To solve this, we find the solution of (2.161) for with $x \leq x'$ and $x > x'$ separately; in each case, the right-hand side is 0, because $\delta(x - x') = 0$ if $x \neq x'$. We will then patch these two solutions at $x = x'$. For $x > x'$ or $x < x'$ (2.161) becomes

$$\frac{d^2f}{dx^2} + \frac{df}{dx} = 0$$

where f is the 'local' solution on either side of x'. Let

$$g = \frac{df}{dx}$$

Then we must solve

$$\frac{dg}{dx} + g = 0 \rightarrow \frac{dg}{dx} = -g$$

The solution is $g(x) = Ce^{-x}$, thus, the solution for f is $f(x) = -Ce^{-x} + D$. Crucially, we have two different solutions on the two side of x':

$$f_L(x) = -C_Le^{-x} + D_L \text{ for } x < x'$$

$$f_R(x) = -C_Re^{-x} + D_R \text{ for } x > x'$$

With boundary condition

$$f_L(0) = 0 \rightarrow -C_L + D_L = 0$$

$$f_R(L) = 0 \rightarrow -C_Re^{-L} + D_R = 0$$

Since this function is continuous, $f_L(x)$ and $f_R(x)$ must match at x', so

$$f_L(x') = f_R(x') \rightarrow -C_Le^{-x'} + D_L = -C_Re^{-x'} + D_R$$

. So far, we have 3 relationships for 4 unknown. We obtain the last relationship by integrating the original ODE (2.161) over the interval $(x' - \epsilon, x' + \epsilon)$ where ϵ is very small. Then

$$\int_{x'-t}^{x'+t} (\frac{d^2G}{dx^2} + \frac{dG}{dx})dx = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x - x')dx$$

Noting that, by definition of the δ-function, $\int_{x'-\epsilon}^{x'+\epsilon}\delta(x-x')dx = 1$, we have

$$[\frac{dG}{dx}]_{x'-t}^{x'+\epsilon} + [G]_{x'-\epsilon}^{x'+\epsilon} = 1$$

We picked f_L and f_R so $G = f_L$ for $x < x'$ and $G = f_R$ for $x > x'$. Because $f_L(x') = f_R(x')$, the Greens function G is continuous at $x = x'$, so when $\epsilon \to 0$, $[G]_{x'-\epsilon}^{x'+\epsilon} = G(x'+\epsilon) - G(x'-\epsilon) \to 0$.

Then, the equation reduces to $[\frac{dG}{dx}]_{x'-\epsilon}^{x'+\epsilon} = 1$, which can also be expressed as

$$\frac{df_R}{dx}|_{x=x'} - \frac{df_L}{dx}|_{x=x'} = 1$$

so

$$C_R e^{-x'} - C_L e^{-x'} = 1 \to C_R - C_L = e^{x'}$$

Thus we constructed 4 equations for the 4 constants C_R, D_R, C_L, D_L. These equations simplify as,

$$C_L = D_L$$

$$C_R e^{-L} = D_R$$

which can be used in the continuity condition to give

$$-C_L e^{-x'} + C_L = -C_R e^{-x'} + C_R e^{-L}$$

with the jump condition being

$$C_R - C_L = e^{x'}$$

The final solution is

$$D_L = C_L = \frac{-1 + e^{x'-L}}{1 - e^{-L}}$$

$$C_R = e^{x'} + C_L = \frac{-1 + e^{x'-L}}{1 - e^{-L}} + e^{x'} = \frac{e^{x'} - 1}{1 - e^{-L}}$$

$$D_R = C_R e^{-L} = \frac{e^{x'} - 1}{1 - e^{-L}} e^{-L}$$

As a result, the function becomes

$$G(x, x') = \begin{cases} -\frac{-1+e^{x'-L}}{1-e^{-L}} e^{-x} + \frac{-1+e^{x'-L}}{1-e^{-L}} & x \le x' \\ -\frac{e^{x'}-1}{1-e^{-L}} e^{-x} + \frac{e^{x'}-1}{1-e^{-L}} e^{-L} & x' \le x \end{cases}$$

Plotting the result of this function, with different x' value, against the Greens function obtained from the eigenvalue method, is shown in Figure 2.10. We can see although the mathematical results obtained using the previous calculation method differ, the visual performance on the image remains the same.

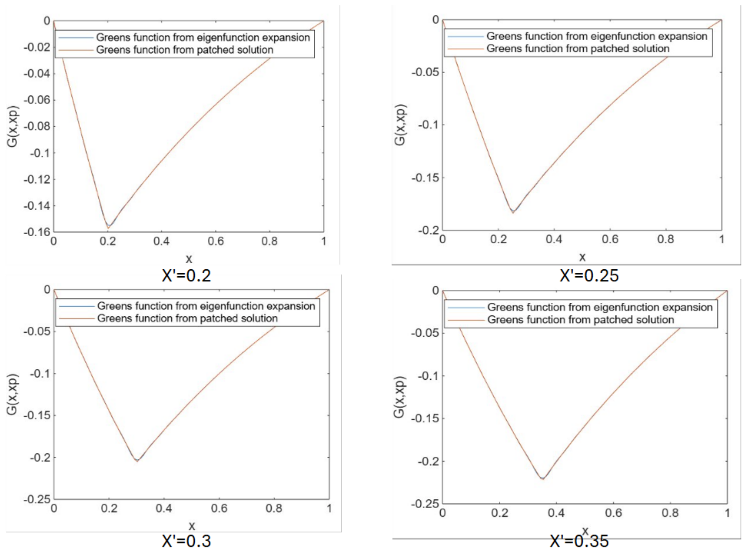

Figure 2.10: This shows the plotted greens function for $x' = 0.2, 0.25, 0.3, 0.35$

2.6.4 Green's functions for time-independent PDEs in bounded domains

Green's functions for PDEs are defined (and found) in similar ways as for ODEs. Notably, given a linear PDE

$$\mathcal{L}f = F(\mathbf{r}) \tag{2.162}$$

with some homogeneous boundary conditions, where $\mathcal{L}$ is a linear partial differential operator in spatial variables only, and F is a function of space, then the Green's function associated with the operator is the solution of

$$\mathcal{L}(G) = \delta(\mathbf{r} - \mathbf{r}') \tag{2.163}$$

with the same boundary conditions and a solution of the forced problem is

$$f(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')F(\mathbf{r}')d^n\mathbf{r}' \tag{2.164}$$

As in the case of the ODEs, the integral covers the spatial domain only, and this only works if the boundary conditions are homogeneous. If the boundary conditions are not homogeneous, then see Haberman for detail.

To find the Green's functions in bounded domains, the most versatile method is that of eigenfunction expansion.

Example: What is the solution of the Poisson equation

$$\nabla^2 f = F(\mathbf{r}) \tag{2.165}$$

on a square plate of size $L \times L$ with Dirichlet boundary conditions? Write it in terms of a Green's function. To solve this problem:

- Choose a coordinate system
- Recall the 2D eigenfunctions of the Laplacian operator on this square plate with Dirichlet boundary conditions.
- Expand f on the basis of these 2D eigenfunctions, and project the equation on that basis
- Solve the resulting algebraic problem, and assemble the solution
- Identify the Green's function by rewriting the solution as in (2.164).

Solution:

We had found the solution to the eigenvalue problem $\nabla^2 f = -\lambda f$ already, and the 2D eigenfunctions were of the form:

$$f_{n,m}(x, y) = \sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{L}) \tag{2.166}$$

$$\text{and } \lambda_{n,m} = (n^2 + m^2)\frac{\pi^2}{L^2} \tag{2.167}$$

The eigenfunctions satisfied the orthogonality relationship

$$\int_0^L \int_0^L f_{n,m}(x, y)f_{n',m'}(x, y)dxdy$$
$$= \delta_{m,m'}\delta_{n,n'}\frac{L^2}{4}$$

We expand the solutions to the forced problem on the basis of these eigenfunctions, so

$$f(x, y) = \sum_{n,m} a_{n,m}f_{n,m}(x, y) \tag{2.168}$$

We also know that the solution can be written as an integral over the Green's function:

$$f(x, y) = \int_0^L \int_0^L G(x, x', y, y')F(x', y')dx'dy' \tag{2.169}$$

If we project our forced PDE onto the eigenfunctions we get

$$\frac{L^2}{4}a_{n,m}(-\lambda_{n,m}) = \int_0^L \int_0^L F(x',y')f_{n,m}(x',y')dx'dy'$$

Solving for $a_{n,m}$ we then get

$$a_{m,n} = -\frac{4}{\lambda_{n,m}L^2}\int_0^L \int_0^L F(x',y')f_{n,m}(x',y')dx'dy' \tag{2.170}$$

So putting this into the solution for $f(x,y)$ we get

$$f(x,y) = -\sum_{n,m} f_{n,m}(x,y)\frac{4}{\lambda_{n,m}L^2}\int_0^L \int_0^L F(x',y')f_{n,m}(x',y')dx'dy' \tag{2.171}$$

Identifying this expression with (2.169) shows that the Green's function is:

$$G(x,x',y,y') = \sum_{n,m} -\frac{4}{\lambda_{m,n}L^2}f_{m,n}(x,y)f_{m,n}(x',y')$$

You may (rightfully) wonder why one would want to solve the Poisson equation in these constrained geometries and boundary conditions, and what is the point of writing the problem using Green's functions rather than solving it directly. The answer is that there are many physical problems where two quantities are related to each other via a Laplacian, as for instance:

- The gravitational potential Φ and mass density ρ: $\nabla^2\Phi = 4\pi G\rho$ where G is the gravitational constant
- The electric potential Φ and the charge density ρ_e: $\nabla^2\Phi = \rho_e/\varepsilon$ (where ε is the permittivity)
- The vorticity ω and stream function ψ in a 2D incompressible fluid: $\nabla^2\psi = \omega$
- The pressure in an incompressible fluid: $\nabla^2 p = F(\mathbf{r})$ where $F(\mathbf{r})$ depends on the local fluid flow.

In most of these problems, we often want to solve for Φ, p or ψ many times with different right-hand sides. By finding the Green's function (with some specified boundary conditions), we can explicitly write the solution to these problems once and for all, and compute the integral in (2.164) to solve for, e.g. Φ, ψ or p.

2.6.5 Green's functions for time-independent PDEs in unbounded domains

The method discussed above works well for PDEs for which there exist a complete basis of eigenfunctions upon which the solution can be expanded. This almost always requires the domain to be bounded (recall that unbounded domains result in singular Sturm-Liouville problems, for which the existence of a basis is not guaranteed).

In unbounded domains, it is better to use a method that is the extension of the 'patching' method introduced in Example 2 of Section 2.6.3, in which we directly solved for the Green's function. In fact, in multiple dimensions the unbounded problem turns out to be a lot easier because of symmetries.

Example 1: Find the Green's function of the 3D Laplacian operator in the infinite domain, solution of

$$\nabla^2 G = \delta(\mathbf{x}-\mathbf{x}') \tag{2.172}$$

where $\mathbf{x}$ and $\mathbf{x}'$ are two position vectors, subject to the condition $G \to 0$ at infinity. To do this:

- Start by showing that if G is the solution of $\nabla^2 G = \delta(\mathbf{x})$ then $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}-\mathbf{x}')$. In other words, we only have to solve the problem once for an impulse at the origin, and simply translate the solution by $\mathbf{x}'$ to get the solution for an impulse at $\mathbf{x}'$.
- Use the fact that $\delta(\mathbf{x})$ is spherically symmetric to show that G only depends on $r = |\mathbf{x}-\mathbf{x}'|$, and thus simplify the Laplacian.

- Solve the equation subject to the condition that G must go to zero as $r \to \infty$.
- Integrate (2.172) over an infinitesimally small sphere centered on the impulse to find the remaining unknown constant in G.
- Use the result to express the electric potential Φ as a function of the charge density field ρ in a 3D universe
- What would have happened for a 2D universe?

Solution: We first prove that G is only a function of $|\mathbf{x}-\mathbf{x}'|$.

To do so, let's first define the solution of the Green's function with impulse forcing at the origin to be $G_0(\mathbf{x})$, so

$$\nabla G_0 = \delta(\mathbf{x}) \tag{2.173}$$

Because the δ-function is 0 everywhere except at the origin, $G_0(\mathbf{x})$ is the solution of a forced problem where the forcing is spherically symmetric. The response will then also be spherically symmetric, and will only depend on $|\mathbf{x}|$. Hence,

$$G_0(\mathbf{x}) = G_0(|\mathbf{x}|)$$

Next, we let $x = \chi - x'$, $y = \eta - y'$ and $z = \zeta - z'$. Then, in these new variables the equation is

$$\frac{\partial^2 G}{\partial \chi^2} + \frac{\partial^2 G}{\partial \eta^2} + \frac{\partial^2 G}{\partial \zeta^2} = \delta(\chi - x')\delta(\eta - y')\delta(\zeta - z') \tag{2.174}$$

This turns out to be the equation for the Greens function centered on $\mathbf{x}' = (x', y', z')$. In other words,

$$G(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x} - \mathbf{x}') = G_0(|\mathbf{x} - \mathbf{x}'|)$$

Let's find what G_0 is. Because the solution is spherically symmetric, we express the Laplacian in spherical coordinates as:

$$\frac{1}{r^2}\frac{\partial}{dr}\left(r^2 \frac{\partial G_0}{\partial r}\right) = \delta(r) \tag{2.175}$$

which, for all $r \neq 0$ is:

$$\frac{1}{r^2}\frac{\partial}{dr}\left(r^2 \frac{\partial G_0}{\partial r}\right) = 0 \tag{2.176}$$

$$\Rightarrow r^2 \frac{\partial G_0}{\partial r} = k \tag{2.177}$$

$$\Rightarrow \frac{\partial G_0}{\partial r} = \frac{k}{r^2} \tag{2.178}$$

$$G_0(r) = -\frac{k}{r} + k' \tag{2.179}$$

We want G_0 to decay as $r \to 0$, so we have that $k' \equiv 0$. To find k, we integrate (2.175) over a tiny sphere of radius ε centered on $r = 0$:

$$\to \int_0^{\varepsilon} \frac{4\pi r^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G_0}{\partial r}\right) dr = \iiint dr(\delta(\mathbf{r})) = \mathbf{1} \tag{2.180}$$

$$\to 4\pi \left[r^2 \frac{\partial G_0}{\partial r}\right]_0^{\varepsilon} = 1 \tag{2.181}$$

$$\to 4\pi\varepsilon \frac{\partial G_0}{\partial r}|_{\varepsilon} = 1 \tag{2.182}$$

$$\to \frac{\partial G_0}{\partial r}|_{\varepsilon} = \frac{1}{4\pi\varepsilon^2} \tag{2.183}$$

$$\to \frac{\partial G_0}{\partial r} = \frac{k}{r^2} \text{ for very small } r \quad \to \quad k = \frac{1}{4\pi} \tag{2.184}$$

$$\to G_0(r) = -\frac{1}{4\pi r} \tag{2.185}$$

We now use this result to express the electric potential Φ as a function of the charge density field ρ in a 3D universe. We start by solving for electric potential:

$$\nabla^2 \Phi_e = \rho(\mathbf{r})e \tag{2.186}$$

$$\rightarrow \Phi_e(\mathbf{r}) = \iiint G(\mathbf{r}, \mathbf{r}')e\rho(\mathbf{r}')d^3\mathbf{r}' \tag{2.187}$$

We write our full greens function:

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{2.188}$$

and substitute this into $\Phi_e(\mathbf{r})$:

$$\Phi_e(\mathbf{r}) = \iiint -\frac{e\rho(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|}d^3\mathbf{r}' \tag{2.189}$$

Note how we leveraged the fact that the domain was unbounded in all directions to find a spherically symmetric Green's function. However, if the domain is only semi-infinite the solution is no longer axially or spherically symmetric, which invalidates this approach. Or maybe not – with an extra trick called *the method of images*.

Example 2: Consider the semi-infinite plane defined by $x > 0$ in Cartesian coordinates. Find the Green's function of the 2D Laplacian operator subject to $G = 0$ at $x = 0$, and $G \rightarrow 0$ as $|\mathbf{r}| \rightarrow \infty$.

- Show that if G_∞ is the Green's function for the 2D Laplacian in the infinite plane (unbounded everywhere), then $G_\infty(x, x', y, y') - G_\infty(x, -x', y, y')$ (with $x' > 0$) is the desired Green's function for this semi-infinite plane problem.
- Interpret physically or graphically why that is the case.

Solution:

2.7 Green's function solutions of forced time-dependent problems

Lecture edited by Dante and Kevin

In the previous lecture, we studied cases of forced second-order linear boundary-value PDEs, i.e. PDEs with no time-dependence or initial conditions. Let us now move to the case of forced time-dependent PDEs. As we shall see, the concepts are similar, but with a few crucial differences.

2.7.1 Green's functions for initial value problem ODEs

Let's begin again with the simpler problem of Green's functions for ODEs, of the form

$$L_t(f) = F(t) \tag{2.190}$$

where L_t is a second-order ordinary linear operator in the single variable t only, and F is some known function of t only. To this 2nd order problem, we need to apply two initial conditions at $t = 0$, usually in the form

$$f(0) = f_0, \frac{df}{dt}(0) = v_0 \tag{2.191}$$

where f_0 and v_0 are constants.

By analogy with the previous lecture, we would like to write the solution to this problem in terms of an integral that involves the Green's function $G(t, t')$, which would be the solution of the ODE with impulse forcing

$$L_t(G) = \delta(t - t') \tag{2.192}$$

Naively, one would guess that the solution should be something like

$$f(t) = \int_0^\infty G(t, t')F(t')dt' \tag{2.193}$$

where the bounds are selected this way because the domain of t is $[0, \infty)$. However, this expression has two fundamental flaws. First, it violates causality: how can the solution know about a forcing that has not happened yet? This suggest that the integral over F should only extend up to $t' = t$. Second, this expression would only work if the equation (and corresponding equation for the Green's function) had homogeneous initial conditions, otherwise the integral itself cannot satisfy these conditions. So how do we modify the problem to allow for non-zero initial conditions? The key is to remember that it is always possible to write the solution as the sum of a general solution of the homogeneous IVP, and a particular solution of the forced IVP:

$$f(t) = f_h(t) + f_p(t) \tag{2.194}$$

That being the case, we can put the responsibility of the initial conditions on the general solution of the homogeneous problem, by requiring that

$$f_h(0) = f_0, \quad \frac{df_h}{dt}(0) = v_0 \tag{2.195}$$

That being dealt with, the particular solution now simply has to satisfy homogeneous initial conditions, and can be written in terms of a Green's function as

$$f_p(t) = \int_0^t G(t, t')F(t')dt' \tag{2.196}$$

where G is the solution of (2.192) with homogeneous initial conditions. It is relatively easy to show that $df_p/dt = 0$ indeed at $t = 0$ under these conditions.

Proof:
Let

$$f_p(t) = \int_0^t G(t, t')F(t')dt' \tag{2.197}$$

If we take the derivative of $f_p(t)$ we have,

$$\frac{df_p}{dt} = \left(\frac{d}{dt}\left(\int_0^t G(t,t')F(t')dt'\right)\right) \tag{2.198}$$

$$= \int_0^t F(t')\frac{\partial G(t,t')}{\partial t}dt' + F(t)G(t,t) \tag{2.199}$$

if we evaluate this at $t = 0$ we have,

$$\frac{df_p}{dt}(0) = \int_0^0 F(t')\frac{\partial G(0,t')}{\partial t}dt' + F(0)G(0,0) = 0 \tag{2.200}$$

if we assume that $F(0) = 0$.

The question then shifts to how to find these Green's functions that are solutions of homogeneous IVPs.

2.7.2 Using Laplace transforms to find Green's function solutions for ODEs

A nice tool for finding solutions of impulse problems for time-dependent ODEs is the Laplace Transform. Let's briefly recall how they work.

The Laplace transform of $f(t)$, denoted in these notes as $\mathcal{T}(f)$ or sometimes $\bar{f}(s)$, defined as:

$$\mathcal{T}(f) = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt \tag{2.201}$$

provided the integral exists. Their inverse is given by the so-called Bromwich integral in the complex plane

$$f(t) = \frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} \bar{f}(z)e^{zt}dz \tag{2.202}$$

where the value of λ is chosen so that $\lambda > 0$, and the line over which the integral is taken lies to the right of all of the singularities of $\bar{f}(z)$. In general, computing the inverse directly can be difficult, and in practice one often relies on tables of 'standard' inverse Laplace Transforms, and a few important properties to find the inverse (or we can use Wolfram Alpha, which is pretty good at finding them).

Important properties of the transform can be found in RHB 13.2.1 and 13.2.2, and those that are particularly useful are:

- Linearity:

$$\mathcal{T}(af + bg) = a\mathcal{T}(f) + b\mathcal{T}(g) \tag{2.203}$$

- Differentiation:

$$\mathcal{T}\left(\frac{df}{dt}\right) = s\mathcal{T}(f) - f(t=0) = s\bar{f}(s) - f(0) \tag{2.204}$$

- Laplace transform of a δ function:

$$\mathcal{T}(\delta(t-a)) = H(a)e^{-sa} \tag{2.205}$$

  where H is the Heaviside function

- Convolution theorem: the inverse transform of $\bar{h}(s) = \bar{f}(s)\bar{g}(s)$ is

$$h(t) = \int_0^t f(u)g(t-u)du = \int_0^t g(u)f(t-u)du \tag{2.206}$$

  where $f(t)$ and $g(t)$ are the inverse Laplace transforms of $\bar{f}(s)$ and $\bar{g}(s)$, respectively.

Let's now see how to use this in practice.

Example 1: Use Laplace transforms to find the solution of

$$\frac{d^2G}{dt^2} + a\frac{dG}{dt} = \delta(t - t') \tag{2.207}$$

that satisfies $G(0, t') = 0$, $G'(0, t') = 0$. Use the result to find the solution of

$$\frac{d^2f}{dt^2} + a\frac{df}{dt} = F(t) \tag{2.208}$$

under the conditions $f(0) = f'(0) = 0$. To do that,

- Take the Laplace Transform of the impulse equation
- Solve the resulting algebraic equation
- Invert the solution to find $G(t, t')$.
- Use that to find $f(t)$.
- Check your answer for a simple function $F(t)$, such as $F(t) = e^{-2t}$ for example.

Solution: Let us find the laplace transform of (2.207).

$$\mathcal{L}\left[\frac{d^2G}{dt^2} + a\frac{dG}{dt}\right] = \mathcal{L}\left[\delta(t - t')\right] \tag{2.209}$$

$$\rightarrow (s^2 + as)\mathcal{L}(G) = \mathcal{L}\left[\delta(t - t')\right. \tag{2.210}$$

$$\rightarrow \mathcal{L}(G) = \mathcal{L}(\delta(t - t'))\mathcal{L}(\xi(t)) \tag{2.211}$$

where

$$\mathcal{L}(\xi(t)) = \frac{1}{s^2 + as} \tag{2.212}$$

Let us perform partial fraction decomposition and find the inverse laplace transform of (2.212).

$$\frac{1}{s^2 + as} = \frac{A}{s} + \frac{B}{s + a} \tag{2.213}$$

$$1 = A(s + a) + Bs \tag{2.214}$$

$$1 = As + Aa + Bs \tag{2.215}$$

$$1 = s(A + B) + As \tag{2.216}$$

$$\implies Aa = 0 \tag{2.217}$$

$$(A + B) = 0 \tag{2.218}$$

$$\begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2.219}$$

$$A = \frac{1}{a} \tag{2.220}$$

$$B = -\frac{1}{a} \tag{2.221}$$

$$\implies \xi(t) = \mathcal{L}^{-1}\left[\frac{\left(\frac{1}{a}\right)}{s} - \frac{\left(\frac{1}{a}\right)}{s + a}\right] \tag{2.222}$$

$$= \frac{1 - e^{-at}}{a} \tag{2.223}$$

(Note that the same result can be obtained from Wolfram Alpha).

The Green's function is thus the convolutionm of $\delta(t-t')$ and $\xi(t)$:

$$G(t,t') = \int_0^t \delta(u-t')\xi(t-u)du \tag{2.224}$$
$$= \int_0^t \delta(u-t')\left(\frac{1-e^{-a(t-u)}}{a}\right)du \tag{2.225}$$
$$= H(t-t')\left(\frac{1-e^{-a(t-t')}}{a}\right) \tag{2.226}$$

where H is the Heaviside function shifted by t'

The solution to any non-homogeneous DE can now be found using this Green's function. Let us find the solution when $F(t) = e^{-2t}$. Notice too that the bounds of integration are from $t' = 0; t' = t$ such that we are only considering when $t' \le t$ so we can drop the Heaviside expression from the integrand.

$$f(t) = \int_0^t G(t,t')F(t')dt' \tag{2.227}$$
$$= \int_0^t \left(\frac{1-e^{-a(t-t')}}{a}\right)e^{-2t'}dt' \tag{2.228}$$
$$= \frac{2-a+ae^{-2t}-2e^{-at}}{4a-2a^2} \quad \text{wolfram alpha} \tag{2.229}$$

Let us verify that this is the solution by plugging back into the original (2.208).

$$\frac{2e^{-2t}-ae^{-at}}{2-a} + a\left(\frac{e^{-at}-e^{-2t}}{2-a}\right) = e^{-2t} \tag{2.230}$$
$$2e^{-2t} - ae^{-at} - ae^{-2t} + ae^{-at} = e^{-2t}(2-a) \tag{2.231}$$
$$e^{-2t}(2-a) = e^{-2t}(2-a)\checkmark \tag{2.232}$$

Example 2: Find the closed-form solution of the forced equation

$$\frac{d^2f}{dt^2} + \omega_0^2 f = F(t) \tag{2.233}$$

with initial conditions

$$f(0) = f_0, \frac{df}{dt}(0) = v_0 \tag{2.234}$$

To do that,

- Follow the same steps as above to find the Green's function with homogeneous conditions, and use that to find the particular solution of this equation.
- Solve the homogeneous problem and apply the initial conditions to it
- Assemble the two pieces to obtain the full solution.

Solution: We begin by taking the Laplace transform of the differential equation and use homogeneous initial conditions.

$$\int_0^\infty \left(\frac{d^2f_p}{dt^2} + \omega_0^2 f_p\right)e^{-st}dt = \int_0^\infty F(t)e^{-st}dt$$
$$s^2\mathcal{L}(f_P) - sf_p(0) - f_p'(0) + \omega_0^2\mathcal{L}(f) = \mathcal{L}(F)$$
$$\mathcal{L}(f_p)\left(s^2+\omega_0^2\right) = \mathcal{L}(F)$$
$$\mathcal{L}(f_p) = \frac{1}{s^2+\omega_0^2}\mathcal{L}(F)$$

Here we must use the convolution theorem for the inverse Laplace transform.

$$f_p(t) = \frac{1}{\omega_0} \int_0^t F(u) \sin(\omega_0(t-u))du$$

In order to find the Green's function, we write $F(t) = \delta(t - t')$.

$$G(t,t') = \frac{1}{\omega_0} \int_0^t \delta(u-t') \sin(\omega_0(t-u))du$$
$$G(t,t') = \frac{1}{\omega_0} H(t-t') \sin(\omega_0(t-t'))$$

This is the Green's function for this forced equation. Next, we solve the homogeneous differential equation with the original initial conditions.

$$\frac{d^2 f_h}{dt^2} = -\omega_0^2 f_h$$
$$f_h = a\cos(\omega_0 t) + b\sin(\omega_0 t)$$
$$f_h(0) = a = f_0, \quad f_h'(0) = \omega_0 b = v_0$$
$$a = f_0, \quad b = \frac{v_0}{\omega_0}$$

The complete solution is then the superposition of these two solutions for a general forcing $F(t)$:

$$f(t) = f_h(t) + \frac{1}{\omega_0} \int_0^\infty H(t-t') \sin(\omega_0(t-t'))F(t')dt'$$
$$f(t) = f_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) + \frac{1}{\omega_0} \int_0^t \sin(\omega_0(t-t'))F(t')dt'$$

Note:

- In both cases, we see that the Green's function is 0 for $t < t'$, which makes physical sense: if the initial conditions are 0, there cannot be a 'response' until the impulse is actually applied.
- In all of these problems, the key rests with being able to find the inverse of the Laplace Transform. This is sometimes easier said than done. Wolfram Alpha can help.
- However, for equations with non-constant coefficients, these transform methods often don't work well, because the transform of the original equation itself is messy.
- Other types of transforms exist that can (sometimes) help, especially if the equation itself derives from a problem in cylindrical or spherical coordinates. (see, e.g. Hankel transforms)
- An alternative method for equations with non-constant coefficients is to use the *method of variation of the coefficients*, also called *method of variation of the parameters*. See RHB section 15.2.4 for detail.

2.7.3 Green's functions for time-dependent PDEs

Lecture edited by Arthur, Henry and Alexandra

We can now combine what we have learned in the previous two lectures to write down the general expression for the solution of a forced time-dependent PDE in terms of a Green's function.

Consider a linear 2nd order PDE (written here in 2D, for simplicity, though the same ideas apply in higher dimensions):

$$\mathcal{L}_t f = \mathcal{L}_x f + F(x,t) \tag{2.235}$$

with initial conditions $f(x,0) = f_0(x)$ (and, if $\mathcal{L}_t$ is second order, an additional condition for the time-derivative of f), and with some homogeneous boundary conditions specified at the edge of the finite interval $[x_a, x_b]$. (As usual, if the boundary conditions are not homogeneous, first rewrite the problem as one that has homogeneous boundary conditions.)

Then, the solution to this equation can be written as

$$f(x,t) = f_h(x,t) + \int_0^t \int_{x_a}^{x_b} F(x',t')G(x,x',t,t')dx'dt' \tag{2.236}$$

where

- $f_h(x,t)$ is the solution to the homogeneous problem $\mathcal{L}_t f = \mathcal{L}_x f$ that satisfies the initial conditions (and homogeneous boundary conditions), and
- $G(x,x',t,t')$ is the solution of

$$\mathcal{L}_t f = \mathcal{L}_x f + \delta(x-x')\delta(t-t') \tag{2.237}$$

To find the Green's functions, we can use a combination of the tools described in the previous sections, and in fact to some extent we have already done so in Sections 2.3.3 and 2.3.4 of these lectures.

Example 1: A rope shaken at one end. In Section 2.3.3 we studied the problem of a rope shaken at one end, and showed that it can be re-cast as the following forced PDE with homogeneous boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t) \tag{2.238}$$

$$u(0,t) = u(L,t) = 0 \tag{2.239}$$

$$u(x,0) = u_0(x), \frac{\partial u}{\partial t}(x,0) = v_0(x) \tag{2.240}$$

where here we have written the forcing term as $F(x,t)$, and the initial conditions as $u_0(x)$ and $v_0(x)$ to give the problem some more generality.

We first find the solutions to the homogeneous problem, and apply the initial conditions, to find $u_h(x,t)$:

Solution: Recall, from an earlier section that the general solution of the 1D wave equation without forcing is given generally by,

$$u(x,t) = \sum_n a_n \cos\left(\frac{n\pi ct}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$
$$+ c_n \sin\left(\frac{n\pi ct}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + d_n \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

What remains is to solve for the boundary conditions and initial conditions. First, we consider the Boundary conditions at $x = 0$ and $x = L$. Since the Dirichlet boundary conditions are homogeneous, the solution is limited to a sine series in x. Thus,

$$u_h(x,t) = \sum_n b_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + d_n \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Now, in order to find the coefficients b_n and d_n we need to consider the initial conditions $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$.

$$u_h(x,0) = \sum_n b_n \sin\left(\frac{n\pi x}{L}\right) = u_0(x)$$
$$\frac{\partial u_h}{\partial t}(x,0) = \sum_n d_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) = v_0(x)$$

Here we take the inner product and rely on the orthogonality of sines to find the coefficients.

$$b_n = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{n\pi x}{L}\right)dx$$
$$d_n = \frac{2}{n\pi c}\int_0^L v_0(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Next, we want to find the particular solution. The spatial eigenmodes of this problem are the standard $\sin(n\pi x/L)$, which form a basis for all functions on $[0,L]$ and after assuming that

$$u_p(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi x}{L}\right) \tag{2.241}$$

and projecting the equation onto each mode, we obtain again the infinite set of ODEs

$$\frac{d^2u_n}{dt^2} + \omega_n^2 u_n = F_n(t) \tag{2.242}$$

with

$$\omega_n = \frac{n\pi c}{L} \tag{2.243}$$
$$F_n(t) = \frac{2}{L}\int_0^L F(x',t)\sin\left(\frac{n\pi x'}{L}\right)dx' \tag{2.244}$$

This now looks a lot like the problem we solved in 2.7.2. Let's finish the problem.

- Find the solution to each ODE problem
- Assemble them to form $u_p(x,t)$
- Express it as (2.164) to find the Green's function for this problem.

Solution: Our ODE is

$$\frac{d^2u_n}{dt^2} + \omega_n^2 u_n = F_n(t) \tag{2.245}$$

with $u_n(0) = 0, u_n'(0) = 0$
Let's use the laplace on both sides

$$s^2L(u_n) - su_n(0) - u_n'(0) + \omega_n^2 L(u_n) = L(F_n) \tag{2.246}$$
$$s^2L(u_n) + \omega_n^2 L(u_n) = L(F_n) \tag{2.247}$$
$$L(u_n) = \frac{L(F_n)}{s^2+\omega_n^2} \tag{2.248}$$

We had already solved this problem a few lectures ago, to find

$$u_n = \frac{1}{\omega_n}\int_0^t F_n(v)\sin(\omega_n(t-v))dv \tag{2.249}$$

Now, if we plug in $\delta(v-t')$ for F, we get the desired Greens function $G_n(t,t')$

$$G_n(t,t') = \frac{1}{\omega_n}H(t-t')\sin(\omega_n(t-t')) \tag{2.250}$$

and we can write u_n as

$$u_n(t) = \int_0^t G_n(t,t')F_n(t')dt' \tag{2.251}$$

We can now substitute $u_n(t)$ into (2.241). We find that

$$u_p(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \int_0^t G_n(t,t')F_n(t')dt' \tag{2.252}$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \int_0^t \frac{1}{\omega_n} \sin(\omega_n(t-t'))\frac{2}{L}\int_0^L F(x',t')\sin\left(\frac{n\pi x'}{L}\right)dx'dt' \tag{2.253}$$

Based on the definition of the Green's function, we know that it is such that

$$u_p = \int_0^t \int_0^L F(x',t')G(x,x',t,t')dx'dt'$$

. We can therefore identify it to be

$$G(x,x',t,t') = \sum_{n=1}^{\infty} \frac{2}{L\omega_n} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)\sin(\omega_n(t-t')) \tag{2.254}$$

so the complete solution of the full problem is:

$$u(x,t) = u_h(x,t) + u_p(x,t) \tag{2.255}$$

$$= \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + d_n \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \tag{2.256}$$

$$+ \int_0^t \int_0^L F(x',t')G(x,x',t,t')dx'dt' \tag{2.257}$$

Example 2: The forced diffusion equation. In Section 2.3.4 we studied a forced diffusion equation of the form

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} + F(x,t) \tag{2.258}$$

$$p(x,0) = 0 \tag{2.259}$$

$$\frac{\partial p}{\partial x}(0,t) = \frac{\partial p}{\partial x}(L,t) = 0 \tag{2.260}$$

Solve this general problem and put it the form (2.235). Hint: in this case you can either use Laplace transforms, or the integrating factor method, to solve the ODEs associated with the problem.

Solution: To solve this equation, we solve for the components of

$$p(x,t) = p_h(x,t) + p_p(x,t)$$

where p_h is the solution to the problem without forcing and p_p is the particular solution. To solve the homogeneous equation, we consider

$$\frac{\partial p_h}{\partial t} = D\frac{\partial^2 p_h}{\partial x^2}$$

$$\frac{\partial p_h}{\partial x}(0,t) = \frac{\partial p_h}{\partial x}(L,t) = 0$$

From previous problems, we have the solution:

$$p_h(x,t) = B_0 + \sum_{n=1} B_n cos(\frac{n\pi x}{L})e^{-D(\frac{n\pi}{L})^2 t}$$

However, if we consider our initial condition, which states $p(x,0) = 0$, we find trivially that the homogeneous solution must be 0. Now to find the particular solution, we require the green's function and thus solve the following:

$$\frac{\partial p_p}{\partial t} = D\frac{\partial^2 p_p}{\partial x^2} + F(x,t)$$

Now, we assume

$$p_p(x,t) = p_0 + \sum_{n=1} \cos(\frac{n\pi x}{L})P_n(t)$$

where $p_0(t)$ and $P_n(t)$ are both functions of time only. To find the equation for p_0, we first integrate (2.261) over the whole domain. The terms in the sum all vanish, and we are left with

$$\begin{aligned}
\int_0^L \frac{\partial p_0}{\partial t}dx &= \int_0^L D\frac{\partial^2 p_0}{\partial x^2}dx + \int_0^L F(x',t)dx' \\
L\frac{\partial p_0}{\partial t}dx &= D\left[\frac{\partial p_0}{\partial x}\bigg|_L - \frac{\partial p_0}{\partial x}\bigg|_0\right] + \int_0^L F(x',t)dx' \\
L\frac{\partial p_0}{\partial t}dx &= \int_0^L F(x',t)dx' \\
\frac{\partial p_0}{\partial t} &= \frac{1}{L}\int_0^L F(x',t)dx' \\
p_0(t) &= \frac{1}{L}\int_0^t\int_0^L F(x',t')dx'dt'
\end{aligned}$$

noting that $p_0(0) = 0$.

To obtain an equation for the $P_n(t)$ functions, we project (2.261) onto our orthogonal basis by multiplying the equation by $\cos(\frac{m\pi x}{L})$ and integrating over the domain $[0, L]$

$$\frac{L}{2}P_n'(t) = -D\frac{L}{2}(\frac{n\pi}{L})^2P_n(t) + \int_0^L F(x',t)\cos(\frac{n\pi x'}{L})dx' \tag{2.261}$$

$$\implies P_n'(t) + D(\frac{n\pi}{L})^2P_n(t) = \frac{2}{L}\int_0^L F(x',t)\cos(\frac{n\pi x'}{L})dx' \tag{2.262}$$

To ease notation, we write:

$$\beta_n(t) = \frac{2}{L}\int_0^L F(x',t)\cos(\frac{n\pi x'}{L})dx' \tag{2.263}$$

We now return to our ODE

$$\implies P_n'(t) + D(\frac{n\pi}{L})^2P_n(t) = \beta_n(t) \tag{2.264}$$

We use the method of integrating factor and find

$$\mu = \exp(D(\frac{n\pi}{L})^2t) \tag{2.265}$$

Which we use to write

$$\frac{d}{dt}[\exp(D(\frac{n\pi}{L})^2t)P_n(t)] = \exp(D(\frac{n\pi}{L})^2t)\beta_n(t) \tag{2.266}$$

We integrate both sides on $[0, t]$

$$\exp(D(\frac{n\pi}{L})^2t)P_n(t) - P_n(0) = \int_0^t \exp(D(\frac{n\pi}{L})^2t')\beta_n(t')dt' \tag{2.267}$$

$$\implies P_n(t) = \int_0^t \exp(D(\frac{n\pi}{L})^2(t'-t))\beta_n(t')dt' \tag{2.268}$$

where we have used the fact that $P_n(0) = 0$. This gives us the particular solution:

$$
\begin{aligned}
p_p(x,t) &= p_0 + \sum_{n=1}^{\infty} \cos(\frac{n\pi x}{L}) P_n(t) \\
p_0(t) &= \frac{1}{L}\int_0^t \int_0^L F(x',t')dx'dt' \\
P_n(t) &= \int_0^t \exp\left(D(\frac{n\pi}{L})^2(t'-t)\right)\beta_n(t')dt' \\
\beta_n(t) &= \frac{2}{L}\int_0^L F(x',t)\cos(\frac{n\pi x'}{L})dx'
\end{aligned}
$$

Now we have the full solution:

$$
\begin{aligned}
p(x,t) = 0 &+ \frac{1}{L}\int_0^t \int_0^L F(x',t') \\
&+ \sum_{n=1}^{\infty} \cos(\frac{n\pi x}{L}) \int_0^t \exp\left(D(\frac{n\pi}{L})^2(t'-t)\right)\left[\frac{2}{L}\int_0^L F(x',t')\cos(\frac{n\pi x'}{L})dx'\right]dt' \\
&= \int_0^L \int_0^t F(x',t')\left[\frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\cos(\frac{n\pi x}{L})\exp\left(D(\frac{n\pi}{L})^2(t'-t)\right)\cos(\frac{n\pi x'}{L})\right]dx'dt'
\end{aligned}
$$

which gives us the Green's function:

$$
G(x,x',t,t') = \frac{1}{L} + \frac{2}{L}\sum_{n=1}\cos(\frac{n\pi x}{L})\exp\left[D(\frac{n\pi}{L})^2(t'-t)\right]\cos(\frac{n\pi x'}{L})
$$

We see, from its construction, that the Green's function in both examples only depends on the operator, and on the boundary conditions applied to it. We had made the same remark in the previous section on time-independent problems. This is a *general* statement, which shows the power of this method: given some operator and boundary conditions, we can find these Green's functions and use them to find the solution to the forced initial value problem for *any* forcing simply by using (2.235). The initial conditions are simply taken care of by the solution to the homogeneous problem, which is generally easy to find (for linear PDEs).

Now, one may again ask *why* bother constructing these Green's functions given that these equations are relatively easy to solve numerically with a PDE solver (cf. what you will/have learned in AM 213B). The answer is that PDE solvers *evolve* the solution forward in time, and therefore have to compute the solution at many intermediate timesteps in order to find the one at time t. Here, we can apply the formula (2.235) directly to find the solution at t without computing any intermediate timesteps.

Chapter 3

Asymptotic analysis

Asymptotic analysis is the theory and associated mathematical tools to deal with mathematical problems that contain parameters or variables that become asymptotically small or asymptotically large. Note that for many purposes, 'asymptotically' just means 'very very'. But mathematically it also has a strict meaning as the limit when a quantity tends to 0 ('asymptotically small'), or a quantity tends to infinity ('asymptotically large').

In many cases, approximations to the solutions of some mathematical problems can be found analytically in these limits, even though the solution cannot be found analytically if the parameters or variables are not asymptotically large or small. That is the power of asymptotic analysis. Similarly, numerical algorithms often break when an equation has asymptotically large or small parameters – because of the resolution needed to solve the problem, or because of the stiffness of the equations. In that respect, asymptotic analysis can often provide answers where numerical methods fail.

3.1 Introductory examples

Lecture edited by Howard and Arthur

In these first introductory examples, we will see cases that *can* be solved analytically exactly, and we will compare them to the solutions that are obtained using very basic tools of asymptotic analysis.

3.1.1 Roots of polynomials

We start by looking at roots of polynomials, because these are easy to solve and have simple interpretations.

Example 1: Consider the quadratic $x^2 + \epsilon x - 1 = 0$

- What are its exact solutions?

Solution: Using the quadratic formula:

$$x = \frac{-\epsilon \pm \sqrt{\epsilon^2 + 4}}{2}$$

Let's now assume we (somehow) forgot how the quadratic formula works, and are unable to derive these solutions analytically. Can we still find *approximate* solutions in the limit $\epsilon \to 0$? The answer is yes, and this is how to do it.

- First, we solve the problem for $\epsilon \equiv 0$, and call these solutions $x_0^\pm$. Here, we get

$$x_0^\pm = \pm 1 \tag{3.1}$$

  Presumably, this is a good approximation to the solutions when ϵ is very, very small – but can we do better?

- Let's now try to find a *better* solution $x_1^\pm = x_0^\pm + \delta_0^\pm$ where $\delta_0^\pm(\epsilon)$ is a small correction, that will presumably depend on ϵ.

  Let's plug $x_1^\pm$ into the original equation:

$$(x_0^\pm + \delta_0^\pm)^2 + \epsilon(x_0^\pm + \delta_0^\pm) - 1 = 0 \tag{3.2}$$

  and simplify, using (3.1):

$$\pm 2\delta_0^\pm + (\delta_0^\pm)^2 \pm \epsilon + \epsilon\delta_0^\pm = 0 \tag{3.3}$$

  Because we assumed $\delta_0^\pm$ is small, we are very tempted to ignore the term in $(\delta_0^\pm)^2$ compared with $\delta_0^\pm$, and we are very tempted to ignore $\epsilon\delta_0^\pm$ compared with ϵ. Let's do it for now, noting that later in this course we will see more formally why that is ok. We then get

$$\pm 2\delta_0^\pm \pm \epsilon \simeq 0 \rightarrow \delta_0^\pm \simeq -\frac{\epsilon}{2} \tag{3.4}$$

  As expected $\delta_0^\pm$ depends on ϵ and as assumed, it is small when ϵ is small. This then shows that after 1 iteration of this method, the 'better' solution is

$$x_1^+ = 1 - \frac{\epsilon}{2} \tag{3.5}$$

$$x_1^- = -1 - \frac{\epsilon}{2} \tag{3.6}$$

- We can continue this iterative process, letting $x_2^\pm = x_1^\pm + \delta_1^\pm$, assuming $\delta_1^\pm$ is small, to get the next correction.

  Solution:

  Plugging in $x_2^\pm = x_1^\pm + \delta_1^\pm$ into the original equation:

$$(x_1^\pm + \delta_1^\pm)^2 + \epsilon(x_1^\pm + \delta_1^\pm) - 1 = (x_1^\pm)^2 + 2x_1^\pm\delta_1^\pm + (\delta_1^\pm)^2 + \epsilon x_1^\pm + \epsilon\delta_1^\pm - 1 = 0$$

$$(x_1^\pm)^2 + 2x_1^\pm\delta_1^\pm + \epsilon x_1^\pm - 1 \simeq 0 \rightarrow \delta_1^\pm = \frac{1 - \epsilon x_1^\pm - (x_1^\pm)^2}{2x_1^\pm}$$

$$\delta_1^+ = \frac{1 - \epsilon(1 - \frac{\epsilon}{2}) - (1 - \frac{\epsilon}{2})^2}{2(1 - \frac{\epsilon}{2})} = \frac{1 - \epsilon + \frac{\epsilon^2}{2} - 1 + \epsilon - \frac{\epsilon^2}{4}}{2 - \epsilon} = \frac{\epsilon^2}{8} + h.o.t$$

$$\delta_1^- = \frac{1 - \epsilon(-1 - \frac{\epsilon}{2}) - (-1 - \frac{\epsilon}{2})^2}{2(-1 - \frac{\epsilon}{2})} = \frac{1 + \epsilon + \frac{\epsilon^2}{2} - 1 - \epsilon - \frac{\epsilon^2}{4}}{-2 - \epsilon} = \frac{-\epsilon^2}{8} + h.o.t$$

$$x_2^+ = x_1^+ + \delta_1^+ =\simeq 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8}$$

$$x_2^- = x_1^- + \delta_1^- \simeq -1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8}$$

This is starting to look like a Taylor expansion of the true solutions $x^\pm(\epsilon)$ in the limit of $\epsilon \rightarrow 0$, and as it turns out, it is – just check using Wolfram Alpha for instance.

In fact, having noticed this, it is rather tempting to assume that we can solve the problem more directly by postulating that the solution can be written as a Taylor expansion, and then simply looking for the coefficients of that expansion. Let's do that, and assume that

$$x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + ... \tag{3.7}$$

Substituting this into the original quadratic we get

$$(a_0 + a_1\epsilon + a_2\epsilon^2 + ...)^2 + \epsilon(a_0 + a_1\epsilon + a_2\epsilon^2 + ...) - 1 = 0 \tag{3.8}$$

Expanding the terms, and rearranging them in powers of ϵ, we get

$$(a_0^2 - 1) + \epsilon(2a_0a_1 + a_0) + \epsilon^2(2a_0a_2 + a_1^2 + a_1) + ... = 0 \tag{3.9}$$

This has to be true for any value of ϵ, and so we find that term by term, we must have

$$a_0^2 - 1 = 0 \tag{3.10}$$
$$2a_1a_0 + a_0 = 0 \tag{3.11}$$
$$2a_0a_2 + a_1^2 + a_1 = 0 \tag{3.12}$$
$$... \tag{3.13}$$

It is easy to check that this indeed recovers the solutions we had earlier.

Check:

$$a_0^2 - 1 = 0 \rightarrow a_0^2 = 1 \Rightarrow a_0 = \pm 1$$

$$2a_1a_0 + a_0 = 0 \rightarrow 2a_1a_0 = -a_0 \Rightarrow a_1 = -\frac{1}{2}$$

$$2a_0a_2 + a_1^2 + a_1 = 0 \rightarrow 2a_0a_2 = \frac{1}{4} \Rightarrow a_2 = \pm\frac{1}{8}$$

We see that postulating a Taylor expansion for the solution is a very nice way of getting an approximate answer very quickly. It also provides a way of bounding the error on the solution: for instance, we know that the error on the truncated expansion $x_1^\pm = \pm 1 - \epsilon/2$ is of the order of ϵ^2, which means that if $\epsilon = 0.1$, the error on the solution will be something like 1 percent.

- Use Wolfram Alpha (or the exact expression) to find the true solution of $x^2 + 0.1x - 1 = 0$ to several decimal places
- Compare it with the approximate solutions $x_1^\pm$. How large are the errors?

Solution: going to the second power of the Taylor Series, we get

$$x_2^- = -1.05125 \tag{3.14}$$
$$x_2^+ = 0.95125 \tag{3.15}$$

Actual solution: $0.95124921972, -1.05124921973$

On the positive root, our difference is -7.8028e-7. On the negative root, our difference is -7.8027e-7. This is actually much smaller than the expected error which is $O(\epsilon^3)$. [Note from Pascale: I'm actually not sure why!!]

The method outlined above works to find approximate solutions to polynomials in many cases, but not always, as we now discover.

Example 2: Consider the polynomial $\epsilon x^2 + x - 1 = 0$

- Use the Taylor expansion method to find asymptotic approximations to the solutions $x(\epsilon)$ of this quadratic
- Why is this method problematic?
- Graph the quadratic using Wolfram Alpha (or any other graphing software) to understand the source of the problem, and find the exact solution of the quadratic (remembering the quadratic formula). What happens to the second root?
- Would the iteration method have found the second root?
- Find the exact solution and its Taylor expansion. Compare with your findings.

Solution:

$$\begin{aligned}\epsilon(a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots)^2 + (a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots) - 1 = 0 \\ \epsilon(a_0^2 + 2a_0a_1\epsilon + a_1^2\epsilon^2 + 2a_0a_2\epsilon^2 + \ldots) + (a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots) - 1 = 0 \\ a_0^2\epsilon + 2a_0a_1\epsilon^2 + \ldots a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots - 1 = 0 \\ a_0 = 1 \\ a_0^2 + a_1 = 0 \\ a_2 + 2a_0a_1 = 0\end{aligned} \tag{3.16}$$

So we get

$$a_0 = 1 \tag{3.17}$$
$$a_1 = -1 \tag{3.18}$$
$$a_2 = 2 \tag{3.19}$$

But notice we only get one solution. It's problematic since a quadratic almost always has two solutions. The iteration method does not work either, and also only gives 1 solution.

Plotting the solution on Wolfram Alpha or Desmos, we find that there are indeed two solutions: a positive one that is close to 1, and a negative one that appears near $-\infty$ when ϵ is very small.

To see this mathematically we compute the exact solution:

$$x = \frac{-1 \pm \sqrt{1+4\epsilon}}{2\epsilon} \tag{3.20}$$
$$x^+ \simeq 1 - \epsilon + 2\epsilon^2 + h.o.t \tag{3.21}$$
$$x^- \simeq -\frac{1}{\epsilon} - 1 + \epsilon + h.o.t \tag{3.22}$$

We only found the positive one using the perturbation method, but now we see that the negative one is indeed very large (the dominant term is $-1/\epsilon$).

This case is an example of a *singular* problem. We will see many more examples (and how to deal with them) later.

Example 3: Consider the polynomial $x^2 - (2+\epsilon)x + 1 = 0$

- Use the Taylor expansion method to find asymptotic approximations to the solutions $x(\epsilon)$ of this quadratic
- Why is this method problematic?
- Try the iteration method instead. What do you find?
- Find the exact solution and Taylor expand it for small ϵ. What do you notice?
- Graph the quadratic using Wolfram Alpha (or any other graphing software) to understand the source of the problem.

Solution: We assume $x = a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots$ then

$$\begin{aligned}(a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots)^2 - (2+\epsilon)(a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots) + 1 = 0 \\ a_0^2 + 2a_0a_1\epsilon + a_1^2\epsilon^2 + 2a_0a_2\epsilon^2 + \ldots - \left(2a_0 + 2a_1\epsilon + \epsilon a_0 + 2a_2\epsilon^2 + a_1\epsilon^2 + \ldots\right) + 1 = 0\end{aligned} \tag{3.23}$$

Matching terms at the lowest order we get

$$a_0^2 - 2a_0 + 1 = 0 \to a_0 = 1$$

Then at the next order

$$2a_0a_1 - 2a_1 - a_0 = 0 \to -1 = 0$$

which is not possible. Clearly, the proposed expansion in ϵ did not work.

Let's use the iterative method to try and find out what happened. The solution for = 0 satisfies

$$x_0^2 - 2x_0 + 1 = 0 \to x_0 = 1$$

Then we let

$$x_1 = x_0 + \delta_0$$

so

$$\begin{aligned}(x_0 + \delta_0)^2 - (2+\epsilon)(x_0 + \delta_0) + 1 = 0\\ x_0^2 + 2\delta_0 x_0 + (\delta_0)^2 - 2x_0 - 2\delta_0 - x_0\epsilon - \delta_0\epsilon + 1 = 0\\ \to 2\delta_0 x_0 + (\delta_0)^2 - 2\delta_0 - x_0\epsilon - \delta_0\epsilon = 0\end{aligned}$$

Using the fact that $x_0 = 1$ we are left with

$$\begin{aligned}2\delta_0 + \delta_0^2 - 2\delta_0 - \epsilon - \epsilon\delta_0 = 0\\ \to \delta_0^2 - \epsilon - \epsilon\delta_0 = 0\end{aligned} \tag{3.24}$$

Given that $\delta_0 \ll x_0 = 1$, we can neglect the last term, which shows that

$$\delta_0^2 - \epsilon = 0 \to \delta_0 = \sqrt{\epsilon} \tag{3.25}$$

This shows that the first correction is proportional to $\sqrt{\epsilon}$ rather than ϵ as in the case 1. Let's check against the exact solution:

$$x = \frac{2 + \epsilon \pm \sqrt{4 + 4\epsilon + \epsilon^2 - 4}}{2} = \frac{2 + \epsilon \pm \sqrt{4\epsilon + \epsilon^2}}{2} \tag{3.26}$$

$$\to = \frac{2 + \epsilon \pm 2\sqrt{\epsilon}\sqrt{1 + \epsilon/4}}{2} \simeq \frac{2 + \epsilon \pm 2\sqrt{\epsilon}(1 + \epsilon/2 + \ldots))}{2} \tag{3.27}$$

This shows that if ϵ is small negative, there is no solution, but more crucially, that the expansion in small ϵ needs to involves powers of $\sqrt{\epsilon}$ rather than powers of only ϵ.

This case, together with the singular problem of Example 2, both demonstrate that the relevant expansion for the solution is not *always* the obvious one, and much of the art of asymptotic analysis is to find *what* the relevant expansion ought to be for a given problem.

3.1.2 Simple ODEs.

Lecture edited by Charlie, Dante, and Kevin

The basic asymptotic techniques (of finding the $\epsilon = 0$ solution, and iteratively correcting it when $\epsilon \neq 0$; or of assuming a Taylor expansion) introduced in the previous lecture to find roots of polynomials also works to find solutions of ordinary differential equations. Similar problems also arise, in some cases. Let's see a few examples.

Example 1: Consider the ODE

$$\frac{df}{dt} = -(t+\epsilon)f \tag{3.28}$$

subject to the initial condition $f(0) = 1$. Let's pretend we are unable to directly solve the equation when $\epsilon > 0$, but that we can solve the equation for $\epsilon = 0$. We take an iterative approach similar to the one we applied for roots of polynomials.

First, let's solve the problem for $\epsilon \equiv 0$ and call the solution $f_0(t)$. We have

$$\frac{df_0}{dt} = -tf_0 \tag{3.29}$$

which has the general solution $f_0(t) = Ke^{-t^2/2}$. To satisfy the initial conditions, we require $K = 1$, so

$$f_0(t) = e^{-t^2/2} \tag{3.30}$$

Next, let's see if we can find a *better* solution for small but non-zero ϵ, calling it $f_1(t) = f_0(t) + \delta_0(t)$, where $\delta_0(t)$ presumably depends on ϵ and is assumed to be small. Note that because the initial condition has already been satisfied by f_0, the new initial condition on δ_0 is $\delta_0(0) = 0$.

Substituting f_1 into the original equation, we have

$$\frac{df_0}{dt} + \frac{d\delta_0}{dt} = -(t+\epsilon)(f_0 + \delta_0) \tag{3.31}$$

and using what we know of f_0, this becomes

$$\frac{d\delta_0}{dt} + \delta_0 t = -\epsilon e^{-t^2/2} - \epsilon\delta_0 \tag{3.32}$$

Assuming δ_0 is small, we can neglect the second term on the right-hand side, and use the integrating factor method for δ_0 to find

$$\delta_0(t) \simeq -\epsilon t e^{-t^2/2} \tag{3.33}$$

where we have used the initial condition on δ_0 to fix the constant of integration. We therefore see that, as hoped, $\delta_0(t)$ is small compared with $f_0(t)$ (and depends on ϵ as suspected). At this iteration, we therefore find that

$$f_1(t) = (1 - \epsilon t)e^{-t^2/2} \tag{3.34}$$

We could continue the process to find the solution at the next iteration, and would find that ultimately the solution takes the form

$$f(t) = a_0(t) + \epsilon a_1(t) + \epsilon^2 a_2(t) + \ldots \tag{3.35}$$

An interesting question is whether / when that series converges – this is something we will address later in this chapter.

In the meantime, and by analogy with the previous section, we see that it may be easier to *postulate* a solution of that form, and then directly find what the functions a_0, a_1, a_2 are. Let's try that approach instead. Substituting (3.42) into (3.41), we get

$$\frac{da_0}{dt} + \epsilon\frac{da_1}{dt} + \epsilon^2\frac{da_2}{dt} + \ldots = -(t+\epsilon)(a_0(t) + \epsilon a_1(t) + \epsilon^2 a_2(t) + \ldots) \tag{3.36}$$

This has to be true for any value of ϵ and any value of t so we can identify term by term,

$$\frac{da_0}{dt} = -ta_0 \tag{3.37}$$

$$\frac{da_1}{dt} = -ta_1 - a_0 \tag{3.38}$$

$$\frac{da_2}{dt} = -ta_2 - a_1 \tag{3.39}$$

$$\ldots \tag{3.40}$$

Conveniently, the equation for a_n only depends on a_{n-1}, so we can solve these like dominoes. It is easy to check that we recover the same solution as in the iterative method, but now we can also quickly get the next order.

Solution: The first function $a_0(t)$ has the solution

$$a_0(t) = C_0 e^{-t^2/2}$$

We apply the initial condition $a_0(0) = 1$, to get

$$a_0(0) = C_0 = 1$$

For each subsequent function, we use the initial condition $a_n(0) = 0$ and proceed with the method of integrating factor. For this differential equation, the integrating factor is $e^{t^2/2}$

$$\begin{aligned}
\frac{da_1}{dt} + ta_1 &= -a_0 = -e^{-t^2/2} \\
e^{t^2/2}\left(\frac{da_1}{dt} + ta_1\right) &= -1 \\
\frac{d}{dt}\left(e^{t^2/2}a_1\right) &= -1 \\
e^{t^2/2}a_1 &= -t + C_1 \\
a_1(t) = -te^{-t^2/2} + C_1 e^{-t^2/2}, &\quad a_1(0) = C_1 = 0 \\
\rightarrow a_1(t) &= -te^{-t^2/2}
\end{aligned}$$

In fact a series can be built using this method of integration, we have,

$$a_n(t) = -e^{-t^2/2}\int_0^t a_{n-1}(t')e^{t'^2/2}dt'$$

And thus,

$$\begin{aligned}
a_2(t) &= \frac{t^2}{2}e^{-t^2/2} \\
a_3(t) &= -\frac{t^3}{6}e^{-t^2/2}
\end{aligned}$$

etc. We assemble the solution:

$$\begin{aligned}
f(t) &= a_0 + \epsilon a_1 + \epsilon^2 a_2 + \ldots \\
f(t) &= e^{-t^2/2} - \epsilon te^{-t^2/2} + \epsilon^2\frac{t^2}{2}e^{-t^2/2} + \ldots = e^{-t^2/2}(1 - \epsilon t + \epsilon\frac{t^2}{2} + \ldots)
\end{aligned}$$

Let us now remember how to solve this equation exactly.

Solution: We solve this equation exactly using separation of variables.

$$\begin{aligned}
\frac{df}{dt} &= -(t+\epsilon)f \\
\frac{df}{f} &= -(t+\epsilon)dt \\
\ln|f| &= -\frac{t^2}{2} - \epsilon t + c \\
f(t) = Ce^{-t^2/2-\epsilon t}, &\quad f(0) = C = 1 \\
f(t) &= e^{-t^2/2-\epsilon t}
\end{aligned}$$

Taylor expanding the solution for fixed t, $\epsilon \to 0$, we see that we recover the asymptotic solution!

Solution:

$$f(t) = e^{-t^2/2}e^{-\epsilon t}$$
$$f(t) = e^{-t^2/2}\left(1 - \epsilon t + \frac{\epsilon^2 t^2}{2} - \frac{\epsilon^3 t^3}{6} + \dots\right)$$

where, $1 - \epsilon t + \epsilon^2 t^2/2 - \epsilon^3 t^3/6 + \dots$ is the Taylor expansion for $e^{-\epsilon t}$ about $t = 0$.

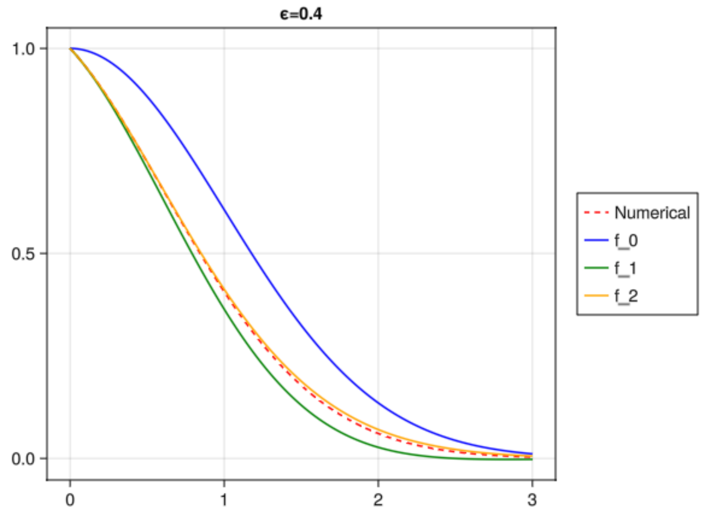

Figure 3.1: Plot of the analytical, numerical, and Taylor-expanded solutions of varying orders.

Graphing the solutions, we also see that keeping even just the first correction is sufficient to capture the effects of ϵ for $\epsilon = 0.1$, say.

In this example, everything worked very well. But by analogy with the roots of polynomials, we can imagine that there are other situations in which problems may arise. Let us see a some examples here.

Example 2: Consider

$$\frac{df}{dt} = -1 - \epsilon f \tag{3.41}$$

subject to the initial condition $f(0) = 0$.

- Assume that the solution takes the form

$$f(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + \dots \tag{3.42}$$

  and find f_0, f_1, f_2, etc..

- Find the exact solution of the equation (numerically, or analytically), and compare its graph to the graph of the approximate solution (keeping successively higher number of terms in the Taylor expansion). What do you notice?

This is an example where the convergence of the asymptotic solution is not ideal. We will see more on these later.

Solution: We try solutions of the following form:

$$f(t) = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots \tag{3.43}$$

so

$$\frac{df_0}{dt} + \epsilon\frac{df_1}{dt} + \epsilon^2\frac{df_2}{dt} = -1 - \epsilon f_0 - \epsilon^2 f_1 - \epsilon^3 f_2 \tag{3.44}$$

Let us match order by order in ϵ to find the ODEs for each f_n in the series

$$\epsilon_0 : \quad \frac{df_0}{dt} = -1 \quad ; \quad f_0(0) = 0 \tag{3.45}$$

$$\implies f_0(t) = -t \tag{3.46}$$

$$\epsilon_1 : \quad \frac{df_1}{dt} = -f_0 \quad ; \quad f_1(0) = 0 \tag{3.47}$$

$$\implies f_1(t) = \frac{t^2}{2} \tag{3.48}$$

$$\epsilon_2 : \quad \frac{df_2}{dt} = -f_1 \quad ; \quad f_2(0) = 0 \tag{3.49}$$

$$\implies f_2(t) = -\frac{t^3}{6} \tag{3.50}$$

thus the approximate function $f(t)$ is

$$f(t) = -t + \epsilon\frac{t^2}{2} - \epsilon^2\frac{t^3}{6} + \dots \tag{3.51}$$

Let us find the analytical solution:

$$\frac{df}{dt} = -1 - \epsilon f \tag{3.52}$$

$$\frac{df}{dt} + \epsilon f = -1 \tag{3.53}$$

$$\int \frac{d}{dt}\left(e^{\epsilon t} f\right) dt = -\int e^{\epsilon t} dt \tag{3.54}$$

$$e^{\epsilon t} f = -\frac{1}{\epsilon} e^{\epsilon t} + c_1 \tag{3.55}$$

$$\implies f(t) = c_1 e^{-\epsilon t} - \frac{1}{\epsilon} \tag{3.56}$$

Applying initial conditions we have:

$$f(0) = c_1 = \frac{1}{\epsilon} \tag{3.57}$$

$$\implies f(t) = \frac{1}{\epsilon}\left(e^{-\epsilon t} - 1\right) \tag{3.58}$$

Figure 3.2 compares the true solution (analytical and numerical) to the approximate solution with different numbers of terms. We see that ultimately the solutions always diverge from the true solution for large enough t, which is not surprising since the approximate solutions are all polynomial in t (and therefore ultimately diverge) while the exact solution converges to $-1/\epsilon$ at large t. The only advantage of keeping more terms is that the solution is valid for longer.

Example 3: Consider the ODE

$$\epsilon\frac{df}{dt} + f = e^{-t} \tag{3.59}$$

with initial condition $f(0) = 2$.

- Try to solve it using the asymptotic tools we have learned (either using iterations, or, using a postulated asymptotic expansion). What happens?
- Now solve it exactly (analytically first, then plot it numerically), and see what the source of the problem is. What happens if we try to Taylor-expand the solution in small ϵ?

Solution:
We start by assuming the solution can be written as a Taylor expansion:

$$f(t) = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots$$

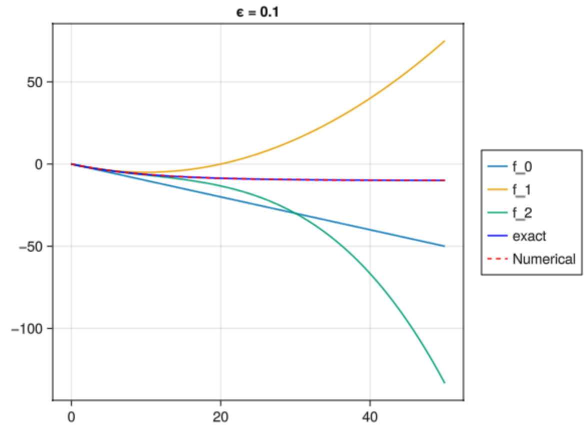

Figure 3.2: Trajectories of approximate solutions to the ODE plotted over the numerical and exact solution. We can see that as t increases, the approximations diverge from the true solution. In fact, the more terms we include the worse our approximation becomes for large t.

which gives the following ODE:

$$\epsilon \frac{df_0}{dt} + \epsilon^2 \frac{df_1}{dt} + \epsilon^3 \frac{df_2}{dt} + ... + f_0 + \epsilon f_1 + \epsilon f_2 + ... = e^{-t}$$

Now matching powers of ϵ, we obtain at lowest order that

$$\epsilon_0 : \quad f_0 = e^{-t} \quad ; \quad f_0(0) = 2$$

We see that the solution for f_0 above is not consistent with the initial conditions of the problem, because $f_0(0) \neq 2$. So, the postulated asymptotic expansion method fails.

The exact solution can be found by $f(t) = f_h(t) + f_p(t)$ where f_h is the homogeneous solution and f_p is the particular solution. This gives:

$$\epsilon \frac{df_h}{dt} + f_h = 0 \implies \frac{df_h}{dt} = -\frac{f_h}{\epsilon}$$

Which can easily be solved for f_h:

$$f_h = c_1 e^{-\frac{t}{\epsilon}}$$

The particular solution satisfies:

$$\epsilon \frac{df_p}{dt} + f_p = e^{-t}$$

We'll use $f_p = c_2 e^{-t}$ as an ansatz, which gives:

$$-\epsilon c_2 e^{-t} + c_2 e^{-t} = e^{-t}$$
$$\implies c_2 = \frac{1}{1-\epsilon}$$

Plugging into the initial equation for $f(t)$ and applying initial conditions gives:

$$f(t) = 2e^{-\frac{t}{\epsilon}} + \frac{1}{1-\epsilon}(e^{-t} - e^{-\frac{t}{\epsilon}})$$

From this we can see that there is *no* way to Taylor expand the solution in the limit of $\epsilon \to 0$, since the term $e^{-t/\epsilon}$ cannot be Taylor-expanded (the derivative of this function with respect to ϵ at $\epsilon = 0$ does not exist).
In this case, we see that the small parameter multiplies the highest-order derivative. This is also a *singular* problem and we see that the 'standard' method does not work in this case.

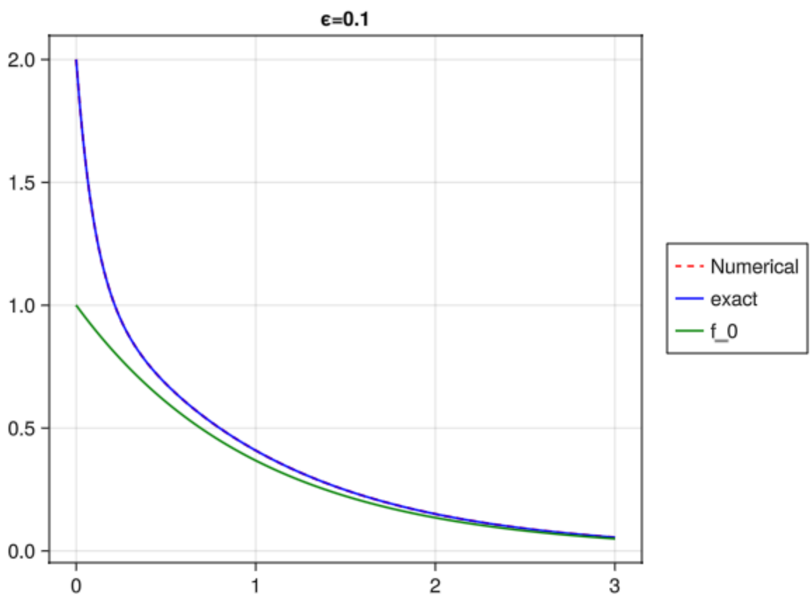

Figure 3.3: Trajectories to the approximate solution, the first asymptotic expansion in green, plotted against the numerical and analytical solution in blue/red. Notice the boundary layer where the exact solution converges to the approximate solution as ϵ becomes small.

3.2 Definitions and tools of asymptotic theory

Lecture edited by Howard, Moein and Alyn

Having seen a few examples how asymptotic theory might work, we now take a step back and introduce important definitions as well as tools that will help us deal more formally with the notion of variables and functions that are either very, very large or very, very small.

3.2.1 The O and o notations

In this chapter, we will rely heavily on the notion of limits at 0 and at infinity, which are tools of basic Calculus that you can review in Chapter 4 of RHB if needed. An important and non-trivial tool for limits that we will use here is l'Hôpital's rule.

l'Hôpital's rule: Given two functions $f(x)$ and $g(x)$ such that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0 \tag{3.60}$$

where a here can either be a finite real number, or $\pm\infty$. If, in addition,

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} \text{ exists,} \tag{3.61}$$

and $g'(x)$ is continuous at $x = a$, with $g'(x) \neq 0 \quad \forall x \neq a$ in the vicinity of a, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{3.62}$$

Note: The condition on g may be a little odd, but it is simply to avoid the pathological case where $g'(x)$ is identically 0 in a finite interval around a. It is fine if $g'(a) = 0$, because if $f'(a) = 0$ as well (which is needed for the limit to exist), we simply apply the rule again. If $f'(a) \neq 0$ then the limit of the ratio of the derivatives does not exist.

Examples:

- $\lim_{x\to\pi/2} \frac{\sin(x-\pi/2)}{x-\pi/2}$

Solution:

$$\lim_{x\to\pi/2} \frac{\sin(x-\pi/2)}{x-\pi/2} = \lim_{x\to\pi/2} \frac{\cos(x-\pi/2)}{1} = \cos(\pi/2-\pi/2) = 1$$

- $\lim_{x\to 0} \frac{1-\cos(x)}{x}$

Solution:

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{1} = \frac{\sin(0)}{1} = 0$$

Having reminded ourselves of this important property of limits, we can now define the O and o notations, which are convenient ways of measuring *how quickly* a function tends to 0 or tends to $\pm\infty$.

Big O notation: Given two functions $f(x)$ and $g(x)$ satisfying (3.60), we say that

$$f(x) = O(g(x)) \text{ as } x \to a \tag{3.63}$$

(where a is either a finite real number or $\pm\infty$) and pronounce it as *f is of the order of g as x tends to a* if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = C \text{ where } 0 < |C| < +\infty \tag{3.64}$$

In practice, this means that f and g both converge to 0 proportionally to one another, that is, at a similar rate.

Little o notation: Given two functions $f(x)$ and $g(x)$ satisfying (3.60), we say that

$$f(x) = o(g(x)) \text{ as } x \to a \tag{3.65}$$

(where a is either a finite real number or $\pm\infty$) and pronounce it as *f is asymptotically smaller than g as x tends to a* if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0 \tag{3.66}$$

In practice, this means that f converges to 0 much faster than g, so that for sufficiently small $|x-a|$, we will have $|f(x)| \ll |g(x)|$.

Examples: Compare

- The functions $f(x) = \sin(x)$ and $g(x) = x$ as $x \to 0$:
  Solution:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \cos(x) = \cos(0) = 1 \therefore f(x) = O(g(x)) \text{ as } x \to 0$$

- The functions $f(x) = 1 - \cos(x)$ and $g(x) = x$ as $x \to 0$
  Solution:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{1} = \sin(0) = 0 \therefore f(x) = o(g(x)) \text{ as } x \to 0$$

- The functions $f(x) = \frac{-3}{x^2-2x+1}$ and $g(x) = \frac{1}{x^2}$ as $x \to +\infty$
  Solution:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to \infty} \frac{-3x^2}{x^2-2x+1} = -3 \therefore f(x) = O(g(x)) \text{ as } x \to \infty$$

Note:

- The function $g(x)$ is called the *gauge function*. It is usually taken to be a function whose properties are well-known as $x \to a$, such as a polynomial function, a exponential function, or a logarithmic function.

Finally, the definitions can now be expanded to various other cases using properties of limits. For instance,

- if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L \neq 0$ then we compare the functions $F(x) = f(x) - L$ and $G(x) = g(x) - L$ instead.
- If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, we can readily expand the definition to say that $f = O(g)$ as $x \to a$ if $\lim_{x\to a} f/g = C$ where $0 < |C| < +\infty$, and, $f = o(g)$ as $x \to a$ if $\lim_{x\to a} f/g = 0$.

Examples:

- The functions $f(x) = x$ and $g(x) = e^x$ as $x \to +\infty$?

Solution:

$$\lim_{x\to \infty} \frac{f(x)}{g(x)} = \lim_{x\to \infty} \frac{x}{e^x} = 0 \Rightarrow x = o(e^x)$$

- The functions $f(x) = x + 1$ and $g(x) = \cos(x)$ as $x \to 0$. Compare $f(x) - L$ with $g(x) - L$.

Solution:

$$\lim_{x\to 0}\frac{f(x)-1}{g(x)-1}=\lim_{x\to 0}\frac{x}{\cos(x)-1}=\infty$$
$$\lim_{x\to 0}\frac{g(x)-1}{f(x)-1}=\lim_{x\to 0}\frac{\cos(x)-1}{x}=0$$
$$\Rightarrow g(x)-1=o(f(x)-1)$$

3.2.2 Convergence of series

Another important tool in asymptotic analysis will be the notion of sequences and series, and convergence of series. Let us start with some basic recap.

Definitions:

- **A sequence** is an infinite list of numbers a_0, a_1, a_2, a_3, ...
- **A series** is the sum of the numbers in the sequence. We distinguish between a finite series,

$$S_N=\sum_{n=0}^{N}a_n \tag{3.67}$$

  and the infinite series

$$S=\lim_{N\to\infty}S_N=\sum_{n=0}^{\infty}a_n \tag{3.68}$$

While the finite series is always defined, the question of whether the infinite series exists or not is both crucial and non-trivial. Here, we state the theorem without proving it.

Convergence of series: To prove that S_N (defined above) converges to a finite limit S, it suffices to show that $\lim_{n\to\infty}|a_{n+1}/a_n|<1$.

Note:

- This implies that a *necessary* condition for the convergence of the series is that $\lim_{n\to\infty}a_n\to 0$.
- However, that condition is *not sufficient.*

Examples:

- Does the series $\sum_{n=0}^{\infty}2^{-n}$ converge? **Solution:**

$$\lim_{n\to\infty}\frac{2^{-(n+1)}}{2^{-n}}=\lim_{n\to\infty}2^{-1}=\frac{1}{2}<1 \quad \checkmark$$

- Does the series $\sum_{n=0}^{\infty}\frac{1}{n+1}$ converge? **Solution:**

$$\lim_{n\to\infty}\frac{(n+2)^{-1}}{(n+1)^{-1}}=\lim_{n\to\infty}\frac{n+1}{n+2}=1 \quad \text{so does not converge}$$

3.2.3 Taylor series

Another important tool of analysis is the notion of a Taylor series.

Theorem: Suppose a function $f(x)$ is differentiable (at least) $N+1$ times at $x = x_0$, then for x sufficiently close to x_0 we can write $f(x)$ as the finite Taylor series

$$f(x) = \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=x_0} + R_N(x) \tag{3.69}$$

where $R_N(x) = O((x-x_0)^{N+1})$ is called the remainder.

Examples: There are several Taylor series of well-known functions near $x = 0$ that are worth knowing by heart (for all the other ones, use Wolfram Alpha):

- $f(x) = e^x \simeq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$
- $f(x) = (1+x)^a \simeq 1 + ax + a(a-1)\frac{x^2}{2!} + a(a-1)(a-2)\frac{x^3}{3!} + ...$
- $f(x) = \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...$
- $f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

Taylor series can be interpreted in two different ways. The simplest is consider N fixed, and to let $x \to x_0$. Then, we know that

$$\lim_{x\to x_0}\left[f(x) - \sum_{n=0}^{N} \frac{(x-x_0)^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=x_0}\right] = \lim_{x\to x_0} R_N(x) = 0 \tag{3.70}$$

because $R_N(x) = O((x-x_0)^{N+1})$ as $x \to x_0$. In other words, the series approximates $f(x)$ near $x = x_0$, and the approximation gets *better and better* as $x \to x_0$.

However, one may also decide to *fix* the value of x near x_0, and ask the question whether the series becomes a *better and better* approximation to $f(x)$ as N increases (assuming that the function $f(x)$ is sufficiently differentiable). That is not guaranteed, and is only true provided

$$\lim_{N\to\infty} R_N(x) = 0 \tag{3.71}$$

for a given fixed value of x. An infinitely differentiable function that satisfies this additional property is called *analytic* at x, and if f is analytic at x, then its Taylor series converges to $f(x)$ as $N \to \infty$.

3.2.4 Asymptotic sequences and asymptotic series

Having defined the notion of sequences and series, and the notion of o, we are now armed with the tools needed to define *asymptotic* sequences and series.

An asymptotic sequence is a sequence of functions $\delta_n(\epsilon)$, i.e. $\{\delta_0(\epsilon), \delta_1(\epsilon), \delta_2(\epsilon), ...\}$, which satisfy the property

$$\delta_{n+1}(\epsilon) = o(\delta_n(\epsilon)) \quad \forall n \text{ as } \epsilon \to 0 \tag{3.72}$$

Examples:

- A common one is $\{1, \epsilon, \epsilon^2, ...\}$
- Another one could be $\{1, \epsilon^{1/2}, \epsilon, \epsilon^{3/2}, ...\}$
- In fact any sequence of the form:

  $$\{1, \epsilon^{\alpha}, \epsilon^{2\alpha}, \epsilon^{3\alpha}, \dots\},$$

  where $\alpha > 0$, forms an asymptotic sequence. This is because

  $$\delta_{n+1}(\epsilon) = \epsilon^{(n+1)\alpha} = o(\epsilon^{n\alpha}) = o(\delta_n(\epsilon)) \quad \text{as } \epsilon \to 0.$$

- Prove that $\{1, \ln(1+\epsilon), \ln(1+\epsilon^2), \ln(1+\epsilon^3), ..$ forms an asymptotic sequence.

To prove that the sequence $\{1, \ln(1+\epsilon), \ln(1+\epsilon^2), \ln(1+\epsilon^3), \dots\}$ forms an asymptotic sequence, we must prove that it satisfies the definition:

$$\delta_{n+1}(\epsilon) = o(\delta_n(\epsilon)) \quad \forall n \text{ as } \epsilon \to 0,$$

which implies

$$\frac{\delta_{n+1}(\epsilon)}{\delta_n(\epsilon)} \to 0 \quad \text{as } \epsilon \to 0.$$

For the given sequence, we have:

$$\delta_0(\epsilon) = 1, \quad \delta_1(\epsilon) = \ln(1+\epsilon), \quad \delta_2(\epsilon) = \ln(1+\epsilon^2), \quad \delta_3(\epsilon) = \ln(1+\epsilon^3), \dots$$

Using the approximation $\ln(1+x) \approx x$ for small x, we find:

$$\frac{\delta_{n+1}(\epsilon)}{\delta_n(\epsilon)} = \frac{\ln(1+\epsilon^{n+1})}{\ln(1+\epsilon^n)} \approx \frac{\epsilon^{n+1}}{\epsilon^n} = \epsilon.$$

As $\epsilon \to 0$, $\frac{\delta_{n+1}(\epsilon)}{\delta_n(\epsilon)} \to 0$. Therefore, $\delta_{n+1}(\epsilon) = o(\delta_n(\epsilon))$, proving that the sequence is asymptotic.

Having defined what an asymptotic sequence is, we note that a function $f(x)$ can (sometimes) be written in the vicinity of x_0 as the *asymptotic series*

$$f(x) = \sum_{n=0}^{N} a_n \delta_n(x - x_0) + R_N(x) \tag{3.73}$$

where R_N is called the remainder.

Example: A Taylor series is a special case of an asymptotic series, using the sequence $\{1, \epsilon, \epsilon^2, ...\}$

Sometimes, a sequence does not contain enough terms, or terms with the right symmetries to approximate a function. For instance, $\{1, \epsilon^2, \epsilon^4, ...\}$ is a legitimate sequence, but it cannot approximate $\sin(x)$ near 0 because it does not have the right symmetries.

Note that the choice of asymptotic sequence to represent a particular function $f(x)$ is not unique – but once a sequence has been chosen, then the coefficients a_n are uniquely defined for that sequence.

Proof:

Let $\{\delta_0(\epsilon), \delta_1(\epsilon), \delta_2(\epsilon) \dots\}$ be an asymptotic sequence and let

$$\begin{aligned} f(\epsilon) &= \sum_{n=0}^{N} a_n \delta_n(\epsilon) + R_N(\epsilon) \\ &= a_0\delta_0(\epsilon) + a_1\delta_1(\epsilon) + a_2\delta_2(\epsilon) + \dots \end{aligned}$$

To find the coefficients a_n, begin by solving for a_0 by computing the limit:

$$a_0 = \lim_{\epsilon \to 0} \frac{f(\epsilon)}{\delta_0(\epsilon)} = \lim_{\epsilon \to 0} \frac{a_0\delta_0(\epsilon) + a_1\delta_1(\epsilon) + a_2\delta_2(\epsilon) + \dots}{\delta_0(\epsilon)} \tag{3.74}$$

$$= \lim_{\epsilon \to 0} \left(a_0 + a_1 \frac{\delta_1(\epsilon)}{\delta_0(\epsilon)} + a_2 \frac{\delta_2(\epsilon)}{\delta_0(\epsilon)} + \dots \right) \tag{3.75}$$

Notice that $\lim_{\epsilon \to 0} \frac{\delta_n(\epsilon)}{\delta_0(\epsilon)} = 0$ for all $n \geq 1$. Thus

$$a_0 = \lim_{\epsilon \to 0} (a_0 + 0 + 0 + \dots) = a_0$$

And for a_1:

$$a_1 = \lim_{\epsilon \to 0} \frac{f(\epsilon) - a_0\delta_0(\epsilon)}{\delta_1(\epsilon)} = \lim_{\epsilon \to 0} \left(a_1 \frac{\delta_1(\epsilon)}{\delta_1(\epsilon)} + a_2 \frac{\delta_2(\epsilon)}{\delta_1(\epsilon)} + \dots \right) = a_1$$

And for a_n:

$$a_n = \lim_{\epsilon \to 0} \frac{f(\epsilon) - \sum_{k=0}^{n-1} a_k\delta_k(\epsilon)}{\delta_n(\epsilon)} = \lim_{\epsilon \to 0} \left(a_n \frac{\delta_n(\epsilon)}{\delta_n(\epsilon)} + a_{n+1} \frac{\delta_{n+1}(\epsilon)}{\delta_n(\epsilon)} + \dots \right) = a_n$$

Thus the coefficients a_n are uniquely defined by the asymptotic sequence.

3.2.5 Uniform and non-uniform convergence

We now arrive at the last and most important part of this section, in which we start to connect these definitions and tools to what we need to solve ODEs using asymptotic analysis.

Suppose we have an ODE (and associated boundary or initial conditions) that contains a small parameter ϵ. In this section we will write the solution to that ODE as $f(x;\epsilon)$, to explicitly remind us that it is a different solution for each value of ϵ. Combining what we saw in the previous lecture, and the definitions we learned today, we may now hypothesize that the asymptotic solution of the ODE can be written as the asymptotic sequence

$$f(x;\epsilon) = \sum_{n=0}^{N} a_n(x)\delta_n(\epsilon) + R_N(x;\epsilon) \tag{3.76}$$

where the $\{\delta_n(\epsilon)\}$ form an asymptotic sequence in the limit of small ϵ. *If the sequence was chosen properly*, then the coefficients a_n, which are now functions of x, are unique.

Examples:

- In the previous lecture, we found that the solution to (3.41) subject to $f(0) = 0$ was $f(t;\epsilon) = -t + \epsilon t^2/2 - \epsilon^2 t^3/6 + ...$ This is based on the asymptotic sequence $\{1, \epsilon, \epsilon^2, ...\}$, with coefficients $a_0(t) = -t$, $a_1(t) = t^2/2$, $a_2(t) = t^3/6$, etc.
- Consider instead a hypothetical situation where we find that the solution to an ODE is the function $g(x;\epsilon) = (1 + \epsilon \sin(x))^{-1}$. Write this function as an asymptotic series for $\epsilon \to 0$? Solution:

  To write this function as an asymptotic series for $\epsilon \to 0$, we use a binomial expansion for $(1+y)^{-1}$, valid for $|y| \ll 1$:
  $$(1 + y)^{-1} = 1 - y + y^2 - y^3 + \dots.$$
  Substituting $y = \epsilon \sin(x)$, we have:
  $$g(x;\epsilon) = 1 - \epsilon \sin(x) + \epsilon^2 \sin^2(x) - \epsilon^3 \sin^3(x) + \dots.$$
  Thus, the asymptotic series for $g(x;\epsilon)$ is:
  $$g(x;\epsilon) = \sum_{n=0}^{\infty} a_n(x)\epsilon^n,$$
  where the coefficients $a_n(x)$ are:
  $$a_0(x) = 1, \quad a_1(x) = -\sin(x), \quad a_2(x) = \sin^2(x), \quad a_3(x) = -\sin^3(x), \dots.$$
  For a truncated series to order N, the remainder term $R_N(x;\epsilon)$ satisfies:
  $$R_N(x;\epsilon) = o(\epsilon^N) \quad \text{as } \epsilon \to 0.$$

The convergence of these two series is very different. In the case of $g(x;\epsilon)$, we can easily verify that as long as $\epsilon < 1$, the series converges as $N \to \infty$ for *all* values of x. This is an example of *uniform* convergence, i.e. the convergence of the series is independent of x.

Proof:

The given function is:
$$g(x;\epsilon) = (1 + \epsilon \sin(x))^{-1}.$$
For $\epsilon < 1$, the series expansion for $g(x;\epsilon)$ is:
$$g(x;\epsilon) = \sum_{n=0}^{\infty} (-1)^n (\epsilon \sin(x))^n.$$

The series converges if the magnitude of the terms decreases to zero as $n \to \infty$. Since $\sin(x)$ is bounded for all x, we have $|\sin(x)| \leq 1$, which implies:

$$|(-1)^n(\epsilon \sin(x))^n| = |\epsilon|^n |\sin(x)|^n \leq |\epsilon|^n.$$

For $\epsilon < 1$, the geometric series $\sum_{n=0}^{\infty} |\epsilon|^n$ converges. Therefore, the series for $g(x;\epsilon)$ converges for all x, and the convergence does not depend on the value of x.

This independence of x is the defining property of *uniform convergence*. Thus, the series for $g(x;\epsilon)$ is uniformly convergent for $\epsilon < 1$.

In the case of $f(t;\epsilon)$, by contrast, the larger t, the more terms need to be kept in order to ensure convergence. This is an example of *non-uniform* convergence, i.e. the convergence of the series depends on t.

Proof:

The given function is:

$$f(t;\epsilon) = -t + \frac{\epsilon t^2}{2} - \frac{\epsilon^2 t^3}{6} + \dots,$$

which can be written as the asymptotic series:

$$f(t;\epsilon) = \sum_{n=0}^{\infty} a_n(t)\epsilon^n,$$

where the coefficients $a_n(t)$ grow with t as t^n.

To analyze the convergence, consider the n-th term of the series:

$$a_n(t)\epsilon^n = \frac{(-1)^{n+1} t^{n+1} \epsilon^n}{(n+1)!}.$$

And now consider the ratio test $\lim_{n\to\infty} |a_{n+1}/a_n|$:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2} t^{n+2} \epsilon^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{(-1)^{n+1} t^{n+1} \epsilon^n} \right| = \lim_{n\to\infty} \left| \frac{t\epsilon}{n+2} \right|$$

For large t, the term t^{n+1} grows rapidly, and although the factorial $(n+1)!$ in the denominator eventually dominates for fixed t, the number of terms required for this dominance increases with t. In other words, as t increases, more terms of the series are needed to approximate $f(t;\epsilon)$ accurately.

Thus, the convergence of the series depends on t: for larger t, the series converges more slowly and requires keeping more terms. This dependence on t implies that the convergence is not uniform.

Therefore, $f(t;\epsilon)$ exhibits *non-uniform* convergence.

Lecture edited by Jeremy and Julian

Definition: An asymptotic expansion is called *uniform* as $\epsilon \to 0$ provided the remainder $R_N(x;\epsilon)$ satisfies

$$|R_N(x;\epsilon)| \le K\delta_{N+1}(\epsilon) \tag{3.77}$$

where K is a constant that is *independent* of x.

Definition: The region of non-uniformity of an asymptotic series is the range of values of the independent variable for which the series is not uniform, ie, where it is not possible to bound the remainder as in (3.77).

In the example of $f(t;\epsilon)$, we see that for $t < 1$, $t^n < 1$, so that

$$R_N(t;\epsilon) \le K_N \epsilon^N \tag{3.78}$$

where K_N is a number that may depend on N, but is independent of t. In other words, the expansion is uniform for $t < 1$. However, when $t > 1$, it is not possible to bound R_N in that way, so the series is non-uniformly convergent for $t > 1$. We conclude that the region of non-uniformity for $f(t;\epsilon)$ is $t > 1$.

In the case of $g(x;\epsilon) = \sum_0^n (-1)^n \sin(x)^n \epsilon^n$, each term $\delta_n(x) = (-1)^n \sin(x)^n$ in the asymptotic expansion is always bounded by ± 1 Therefore, there is no region of non-uniformity, the series is uniformly convergent for $x \in \mathbb{R}$.

Note:

- It should be fairly clear that uniform convergence is strongly preferred over non-uniform convergence. For a uniformly-converging series, we can keep a fixed number of terms and be guaranteed that the order of the error is the same for *any* value of the independent variable. For a non-uniformly converging series, this is not the case.
- Non-uniform convergence is *not* the same as lack of convergence. In fact, in all of the examples above the series converges, but the rate of convergence is quite different for uniform vs. non-uniform series.

3.2.6 Sources of non-uniformity

When solving ODEs using asymptotic methods, there are two cases that often naturally give rise to non-uniform expansions:

- the independent variable is not bounded (usually the case for initial-value problems, where $t \to \infty$).
- the governing equation is singular (i.e. the small parameter ϵ multiplies the highest-order derivative).

We now look at 3 examples, and see how in each case the 'naive' asymptotic method gives rise to a non-uniform expansion.

Example 1: In Example 2 of Section 3.1.2, we studied the IVP ODE

$$\frac{df}{dt} = -1 - \epsilon f \text{ with } f(0) = 0 \tag{3.79}$$

and found that the asymptotic expansion was $f(t) = -t + \epsilon t^2/2 - \epsilon^2 t^3/6 + ...$

- What is the region of uniform convergence for this solution?
- Plot a few representative asymptotic solutions keeping different number of terms in the expansion, and compare it with the true solution. Note how non-uniform convergence requires keeping increasingly more terms to get a meaningful approximation to the true solution.

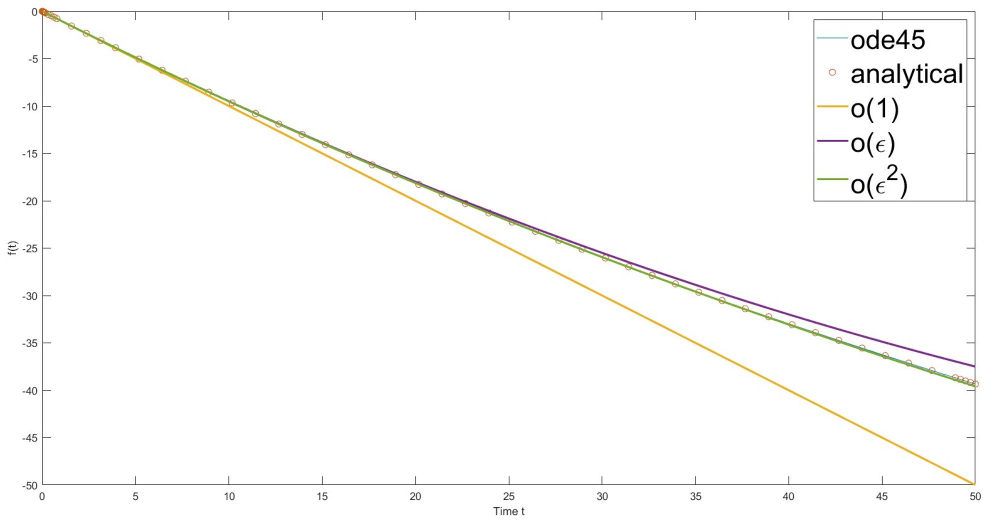

Figure 3.4: Plotting different approximations to (3.55) in orders of ϵ. We need more terms to approach good convergence than just 1 or 2.

Solution: The region of uniformity of $f(t)$ is given by values of t for which the remainder is bounded no matter the order of the terms added. This occurs when $t < 1$.

Example 2: Let us consider another IVP ODE, the nonlinear oscillator called the *Duffing equation*, in which

$$\frac{d^2 f}{dt^2} = -f(1 + \epsilon f^2) \tag{3.80}$$

This equation commonly appears in mechanical problems or electrical circuits. We also have initial conditions:

$$f(0) = h_0, \frac{df}{dt} = 0 \tag{3.81}$$

Let's start by assuming that solutions exist, that have the following asymptotic expansion:

$$f(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f^2(t)... \tag{3.82}$$

and use this ansatz into the governing equation.

- Match terms order by order, to obtain a series of equations and corresponding boundary conditions for f_0, f_1, etc..
- Solve the equations for f_0 and f_1
- Show that this expansion is not uniform.
- Plot the exact solution against the approximate solution to see what is happening.

Solution: Plugging the expansion into our ODE:

$$(\frac{d^2 f_0}{dt^2} + \epsilon \frac{d^2 f_1}{dt^2} + \epsilon^2 \frac{d^2 f_2}{dt^2} + ...) = -(f_0 + \epsilon f_1 + \epsilon^2 f_2)(1 + \epsilon(f_0 + \epsilon f_1 + \epsilon^2 f_2 + ...)^2) \tag{3.83}$$

Matching powers of ϵ, we have at order ϵ^0:

$$\frac{d^2 f_0}{dt^2} = -f_0 \tag{3.84}$$

$$\rightarrow f_0(t) = c_1 \cos(t) + c_2 \sin(t) \tag{3.85}$$

$$\rightarrow f_0(t) = h_0 \cos(t) \tag{3.86}$$

after applying the initial conditions $f_0(0) = h_0, df_0/dt(0) = 0$. At the next order ($O(\epsilon)$):

$$\frac{d^2 f_1}{dt^2} = -f_1 - f_0^3 \rightarrow \frac{d^2 f_1}{dt^2} + f_1 = -h_0^3 \cos^3(t) = -\frac{h_0^3}{4}(3\cos(t) + \cos(3t)) \quad (3.87)$$

$$f_1 = c_1 \cos(t) + c_2 \sin(t) - \frac{3h_0^3}{8} t \cdot \sin(t) + \frac{h_0^3}{32} \cos(3t) \quad (3.88)$$

using e.g. Wolfram Alpha. We see that the term in $t\sin(t)$ grows indefinitely as $t \to \infty$, which makes our approximation non-uniform. The source of the problem is the term in $\cos(t)$ in (3.87), because that is the term that leads to particular solution proportional to $t\sin(t)$.

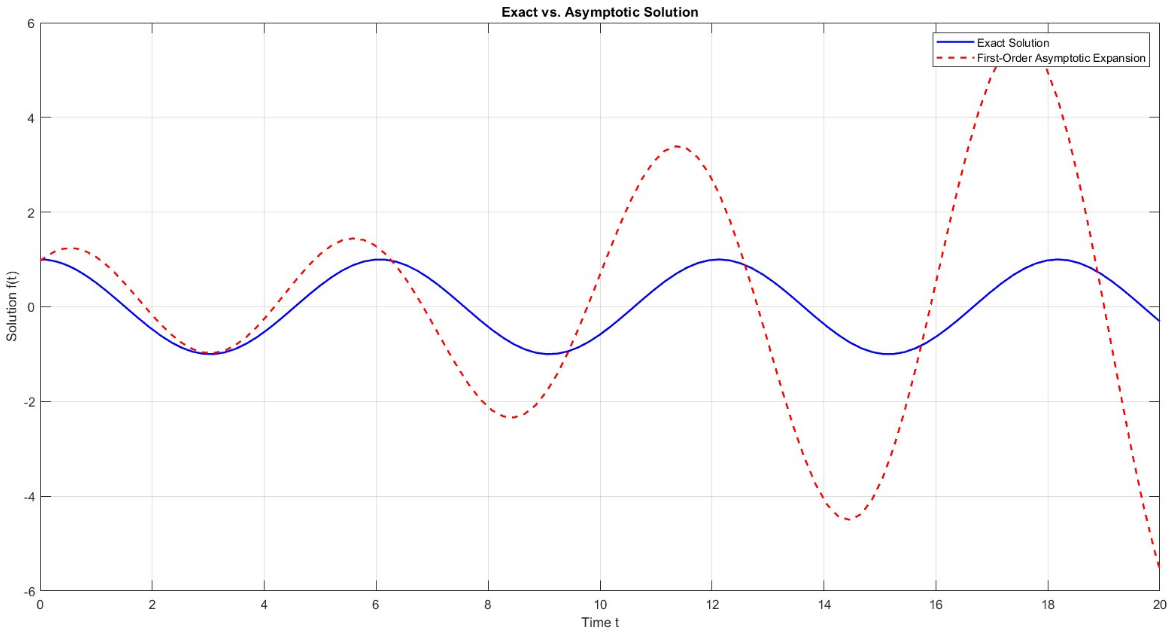

Figure 3.5: Plotting different approximations of f in orders of ϵ. We see that our approximation blows up.

Example 3: Consider the two-point boundary value ODE

$$\epsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} = 2x + 1 \quad (3.89)$$

with $f(0) = 1$, $f(1) = 4$.

- Try to solve it using standard asymptotic techniques (conclude they do not work).
- Find the exact solution.
- Try to Taylor expand it in the limit of $\epsilon \to 0$. What happens? For which values of x does the Taylor-expansion exist?

Solution: Attempting a direct asymptotic expansion $f(x) = f_0(x) + \epsilon f_1(x) + \ldots$ and dropping to lowest-order terms in ϵ, we get:

$$\epsilon\left(\frac{d^2 f_0}{dx^2} + \epsilon\frac{d^2 f_1}{dx^2} + \epsilon^2\frac{d^2 f_2}{dx^2} + \ldots\right) + \left(\frac{df_0}{dx} + \epsilon\frac{df_1}{dx} + \epsilon^2\frac{df_2}{dx} + \ldots\right) = 2x + 1$$
$$\implies \frac{df_0}{dx} = 2x + 1.$$

While this expression may be directly integrated to $x^2 + x + c$, the problem is that this cannot fit both boundary conditions $f(0) = 1$ and $f(1) = 4$ (i.e. our integration constant c cannot be made to satisfy both), and therefore a direct asymptotic expansion will fail here.

On the other hand, it is easy to find the exact solution for this problem. We let $g(x) = \frac{df}{dx}$ so

$$\epsilon\frac{dg}{dx} + g = 2x + 1$$
$$\implies \frac{dg}{dx} + \frac{g}{\epsilon} = \frac{2x + 1}{\epsilon}$$

And now this is in the form of a first-order ODE (which can always be solved via the integrating-factor method):

$$\mu(x) = e^{\int \frac{dx}{\epsilon}} = e^{\frac{x}{\epsilon}}$$
$$\implies \frac{d}{dx}(e^{\frac{x}{\epsilon}} g) = \frac{e^{\frac{x}{\epsilon}}(2x + 1)}{\epsilon}$$
$$\implies e^{\frac{x}{\epsilon}} g(x) - e^0 g(0) = e^{\frac{x}{\epsilon}}(2x - 2\epsilon + 1) + 2\epsilon - 1$$
$$\implies g(x) = (2x - 2\epsilon + 1) + e^{-\frac{x}{\epsilon}}(2\epsilon - 1 + g(0)).$$

Integrating again (where recall $g(x) = \frac{df}{dx} \implies f(x) = \int g(x)dx$, we have an expression involving terms of $e^{\frac{x}{\epsilon}}$:

$$f(x) = \int g(x)dx = -\epsilon(g(0) + 2\epsilon - 1)e^{\frac{-x}{\epsilon}} + x^2 - 2x\epsilon + x + C, \tag{3.90}$$

where C is some integration constant. Both C and $g(0)$ can be found by fitting the boundary conditions.

We see that $f(x)$ contains terms involving $e^{\frac{x}{\epsilon}}$. Consider the Taylor expansion of this expression:

$$e^{\frac{x}{\epsilon}} = \sum_{n=0}^{\infty} \frac{x^n}{\epsilon^n n!}. \tag{3.91}$$

This Taylor expansion does not converge, but in fact diverges as we take the limit $\epsilon \to 0$.

In other words: when we attempted to first implement a direct perturbation expansion to our two-point BVP described by equation (3.91), these expansions rely on the ability to converge to something sensible.

The fact our Taylor expansion of the exact solution however, involves terms that blow up rather than converge as our ϵ parameter becomes increasingly small indicates that our perturbation expansion has nothing it can converge to.

3.3 Rescaling the independent variable

Lecture edited by Henry, Alex, and Arthur

In this section, we now study a first class of techniques to deal with non-uniformity in nonlinear oscillators (such as the Duffing oscillator of the previous section). The techniques we will see here only work when the primary effect of the (weak) nonlinearity of the problem is to cause a change in the frequency of the oscillator, in which case a simple rescaling of the independent variable solves the problem. These types of methods completely fail otherwise. So how do we know whether the nonlinearity merely affects the period of oscillation and not something else? It's easy – just integrate the problem numerically for a few values of ϵ to see what the solutions look like. In asymptotic analysis, it's important to always take a peak at the solution (if possible) to understand its behavior before diving into the analytical calculations, otherwise we can waste a lot of time.

Examples: Plot a few numerical solutions of the following oscillators to see which ones may qualify as 'oscillators where the nonlinearities mostly cause a change of oscillation frequency'.

- $\frac{d^2f}{dt^2} = -f(1+\epsilon f^2)$ with $f(0) = 1$, $df/dt(0) = 0$.
- $\frac{d^2f}{dt^2} = -f + \epsilon f\left[1 - 2\left(\frac{df}{dt}\right)^2\right]$ with $f(0) = 1$, $df/dt(0) = 0$.
- $\frac{d^2f}{dt^2} = -f + \epsilon(1-f^2)\frac{df}{dt}$ with $f(0) = 1$, $df/dt(0) = 0$.

Figures: We plot solutions to these DEs in figures (3.6), (3.7, and 3.8).

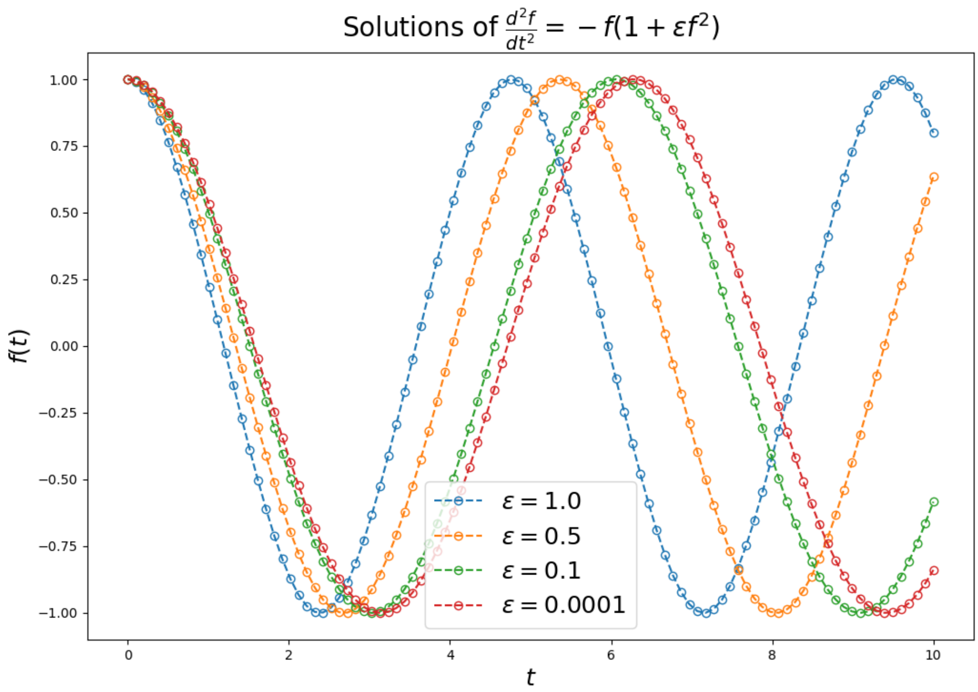

Figure 3.6: A few numerical solutions to $\frac{d^2f}{dt^2} = -f(1+\epsilon f^2)$ with $f(0) = 1$, $df/dt(0) = 0$ for different values of ϵ. The ϵ parameter appears to be changing the oscillation frequency.

3.3.1 The method of strained coordinates (the Linsted-Poincaré technique)

The Linsted-Poincaré technique is great for finding a uniform expansion for nonlinear oscillators where we *know for sure* that the nonlinearity simply causes a change in the period of the oscillator, as in Examples 1 and 2 above.

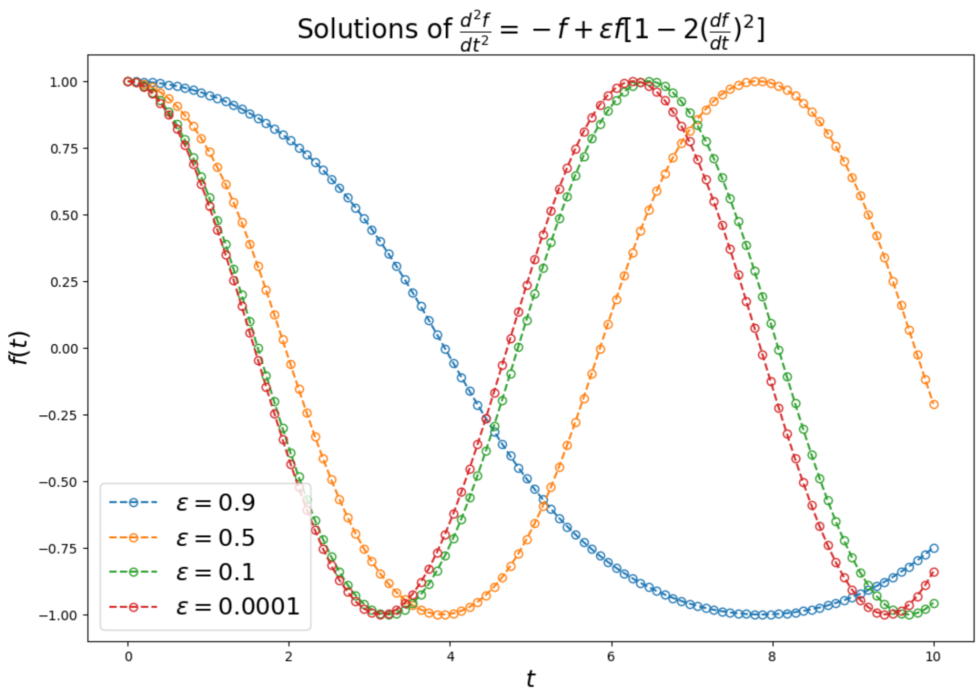

Figure 3.7: A few numerical solutions to $\frac{d^2f}{dt^2} = -f + \epsilon f\left[1 - 2\left(\frac{df}{dt}\right)^2\right]$ with $f(0) = 1$, $df/dt(0) = 0$ for different values of ϵ. The ϵ parameter appears to be changing the oscillation frequency.

In conjunction with proposing the usual asymptotic expansion for $f(t)$ as

$$f(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + ... \tag{3.92}$$

we now *also* propose the following change of coordinates:

$$\tau = t(1 + a_1\epsilon + a_2\epsilon^2 + ...) \tag{3.93}$$

(where the a_n are just constants). This stretches or squeezes the variable τ compared to t, in a manner that depends on ϵ (to be determined).

With this ansatz

$$\frac{df}{dt} = \frac{d\tau}{dt}\frac{df}{d\tau} = (1 + a_1\epsilon + a_2\epsilon^2 + ...)\frac{d}{d\tau}(f_0 + \epsilon f_1 + \epsilon^2 f_2 + ...) \tag{3.94}$$

and similarly for second-order derivatives:

$$\frac{d^2f}{dt^2} = (1 + a_1\epsilon + a_2\epsilon^2 + ...)^2\frac{d^2}{d\tau^2}(f_0 + \epsilon f_1 + \epsilon^2 f_2 + ...) \tag{3.95}$$

The idea will be to pick the constants a_n to eliminate any non-uniform term that appears in the expansion. Let's see how it works on two examples.

Example 1: The Duffing oscillator. We start from (3.80) with initial conditions (3.81) (and $h_0 = 1$, say). After substituting the ansatz for t and for f, we get

$$(1 + a_1\epsilon + O(\epsilon^2))^2\frac{d^2}{d\tau^2}(f_0 + \epsilon f_1 + O(\epsilon^2)) = -(f_0 + \epsilon f_1 + O(\epsilon^2))(1 + \epsilon(f_0 + \epsilon f_1 + O(\epsilon^2))^2) \tag{3.96}$$

with boundary conditions

$$f(0) = f_0(0) + \epsilon f_1(0) + ... \tag{3.97}$$

$$(1 + a_1\epsilon + ...)\frac{d}{d\tau}(f_0 + \epsilon f_1 + ...) = 0 \tag{3.98}$$

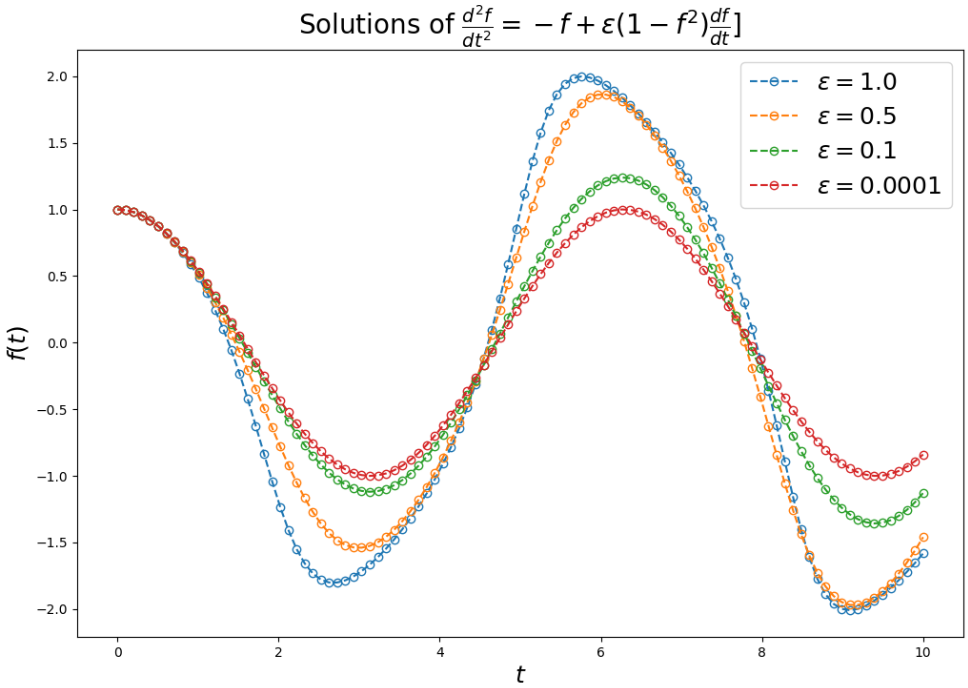

Figure 3.8: A few numerical solutions to $\frac{d^2f}{dt^2} = -f + \epsilon(1-f^2)\frac{df}{dt}$ with $f(0) = 1$, $df/dt(0) = 0$ for different values of ϵ. The ϵ parameter appears to be changing the amplitude of the waves.

Let's collect all the terms order by order. At the lowest order $O(1)$ we have

$$\frac{d^2 f_0}{d\tau^2} = -f_0 \tag{3.99}$$

to which we apply $O(1)$ boundary conditions $f_0(0) = 1$, $df_0/d\tau = 0$. This has the solution

$$f_0(\tau) = \cos(\tau) \tag{3.100}$$

At the next order $O(\epsilon)$ we have

$$\frac{d^2 f_1}{d\tau^2} + 2a_1\frac{d^2 f_0}{d\tau^2} = -f_1 - f_0^3 \tag{3.101}$$

to which we need to apply the initial conditions $f_1(0) = 0$ (because the f_0 function takes care of the 1) and $a_1 df_0/d\tau + df_1/d\tau = df_1/d\tau = 0$.

So far, this looks like what we did naively in Section 3.2 that ultimately led to the non-uniform expansion, except for the factor containing a_1, which is new. Substituting the $O(1)$ solution, this equation becomes

$$\frac{d^2 f_1}{d\tau^2} - 2a_1\cos(\tau) = -f_1 - \frac{1}{4}(\cos(3\tau) + 3\cos(\tau)) \tag{3.102}$$

When solving this equation, we know that the terms in $\cos(\tau)$ are the source of non-uniformity, so we simply pick a_1 to get rid of them:

$$a_1 = \frac{3}{8} \tag{3.103}$$

This means that

$$\tau = \left(1 + \frac{3}{8}\epsilon\right)t \tag{3.104}$$

so that

$$f_0(t) = \cos\left[\left(1 + \frac{3}{8}\epsilon\right)t\right] \tag{3.105}$$

Meanwhile, the equation for f_1 becomes

$$\frac{d^2 f_1}{d\tau^2} = -f_1 - \frac{1}{4}\cos(3\tau) \tag{3.106}$$

which has a solution of the form

$$f_1(\tau) = a\cos(\tau) + b\sin(\tau) + c\cos(3\tau) \tag{3.107}$$

which is nicely bounded as $t \to \infty$. In other words, the expansion is uniform, as desired. We therefore have

$$f(t) = \cos\left[\left(1 + \frac{3}{8}\epsilon\right)t\right] + O(\epsilon) \tag{3.108}$$

Note:

- By construction, this method gets rid of the non-uniform terms that plagued the naive expansion we had obtained in Section 3.2. Accordingly, the amplitude of the uniform asymptotic solution is no longer growing with t as the non-uniform one did.
- In fact, plotting the solution shows that it is remarkably close to the exact solution, even when ϵ is not that small.
- To get the $O(1)$ term in the expansion for $f(t)$, we had to go to the $O(\epsilon)$ equation to find what is the correct stretching factor for $t(\tau)$.
- If we had wanted to get the second term in the expansion for $f(t)$, we would have needed to go to the $O(\epsilon^2)$ equation for f, to get the next term in the expansion for $t(\tau)$. See Bush textbook section 3.2.
- Many of the steps of the Linstead-Poincaré technique are somewhat repetitive of the steps we already did when we discovered that the naive approach leads to a non-uniform expansion. The technique introduced in the next section leverages this to save time (see below).

Example 2: Consider the second oscillator of the list of examples provided earlier.

- Propose expansion (3.92) and (3.93) for f and τ, and find the $O(1)$ and $O(\epsilon)$ equations and boundary conditions.
- Solve the $O(1)$ problem, and substitute into the $O(\epsilon)$ equation. Select the first unknown coefficient of the τ expansion to eliminate the source of non-uniform terms.
- Construct the final solution, and compare it with the exact solution.

Solution:
We consider the following problem:

$$\begin{aligned}
\frac{\partial^2 f}{\partial t^2} &= -f + \epsilon f[1 - 2\frac{\partial f}{\partial t}^2] \\
f(0) &= 1 \\
\frac{\partial f}{\partial t}|_0 &= 0
\end{aligned}$$

We use the asymptotic expansion for $f(t) and a proposed change of coordinates \tau$,

$$\begin{aligned}
f(t) &= f_0 + \epsilon f_1 + ... \\
\tau &= t(1 + a_1\epsilon + ...)
\end{aligned}$$

We also consider the derivatives,

$$\begin{aligned}
\frac{df}{dt} &= \frac{d\tau}{dt}\frac{df}{d\tau} \\
&= (1 + a_1\epsilon + a_2\epsilon^2 + ...)\frac{d}{d\tau}(f_0 + \epsilon f_1 + \epsilon^2 f_2 + ...) \\
\frac{d^2 f}{dt^2} &= (1 + a_1\epsilon + a_2\epsilon^2 + ...)^2\frac{d^2}{d\tau^2}(f_0 + \epsilon f_1 + \epsilon^2 f_2 + ...)
\end{aligned}$$

Now we substitute each of these terms and find,

$$(1+a_1\epsilon+...)^2\frac{d^2}{d\tau^2}(f_0+\epsilon f_1+...)=-(f_0+\epsilon f_1+...)+\epsilon(f_0+...)\left[1-2\left((1+...)\frac{d}{d\tau}(f_0+...)\right)^2\right]$$

Now we consider orders of epsilon,

$$\begin{aligned}
O(1):&\frac{\partial^2 f_0}{\partial\tau^2}=-f_0,\\
&f_0(0)=1\\
&\frac{\partial f_0}{\partial\tau}|_0=0\\
O(\epsilon):&\frac{\partial^2 f_1}{\partial\tau}+2a_1\frac{\partial^2 f_0}{\partial\tau^2}=-f_1+f_0\left[1-2\left(\frac{\partial f_0}{\partial\tau}\right)^2\right]\\
&f_1(0)=0\\
&\frac{\partial f_1}{\partial\tau}|_0=0
\end{aligned}$$

For the first ODE, we find the general solution $f_0(\tau)=c_0\sin(\tau)+c_1\cos(\tau)$, and applying initial conditions we find $f_0(\tau)=\cos(\tau)$. Now we substitute f_0 into our equation for $O(\epsilon)$ and obtain,

$$\begin{aligned}
\frac{\partial^2 f_1}{\partial\tau}+2a_1\frac{\partial^2 f_0}{\partial\tau^2}&=-f_1+f_0\left[1-2\left(\frac{\partial f_0}{\partial\tau}\right)^2\right]\\
\frac{\partial^2 f_1}{\partial\tau}+f_1&=-2a_1\frac{\partial^2 f_0}{\partial\tau^2}+f_0-2f_0\left(\frac{\partial f_0}{\partial\tau}\right)^2\\
&=2a_1\cos(\tau)+\cos(\tau)-2\cos(\tau)\sin^2(\tau)\\
&=2a_1\cos(\tau)+\cos(\tau)(1-2\sin^2(\tau)\\
&=2a_1\cos(\tau)+\frac{1}{2}(\cos(\tau)+\cos(3\tau))
\end{aligned}$$

Now to remove the secular terms in $\cos(\tau)$, we set $0=2a_1+1/2 \implies a_1=-1/4$.
Now we may form the first order solution:

$$\begin{aligned}
f_0(t)&=\cos(\tau) &(3.109)\\
&=\cos(t(1+\epsilon a_1)) &(3.110)\\
&=\cos(t-\epsilon t/4) &(3.111)\\
&=\cos(t(1-\frac{\epsilon}{4})) &(3.112)
\end{aligned}$$

We compare this first order solution to the exact/numeric solution in figure (3.9).

3.3.2 Renormalization

A second technique called *renormalization* also exists for these kinds of problems, which works very well assuming you have already done the work of finding a non-uniform asymptotic expansion of the solution, as we did in Section 3.2. This avoids having to re-do some of the calculations again, and very quickly reveals the correct stretched coordinate with little additional work. Let's see how it works through examples.

Example 1: In Section 3.2, we studied the Duffing oscillator (3.80) with boundary conditions (3.81). We found that a non-uniform expansion to order ϵ of the solution was

$$f(t)=f_0(t)+\epsilon f_1(t)+O(\epsilon^2)=\cos(t)+\frac{\epsilon}{8}\left(\frac{1}{4}\cos(3t)-\frac{1}{4}\cos(t)-3t\sin(t)\right)+O(\epsilon^2) \quad (3.113)$$

As for the Linsted-Poincaré technique, the idea of renormalization is to note that t is no longer the correct independent variable for the solution when $\epsilon\neq 0$. Instead, the relevant independent variable

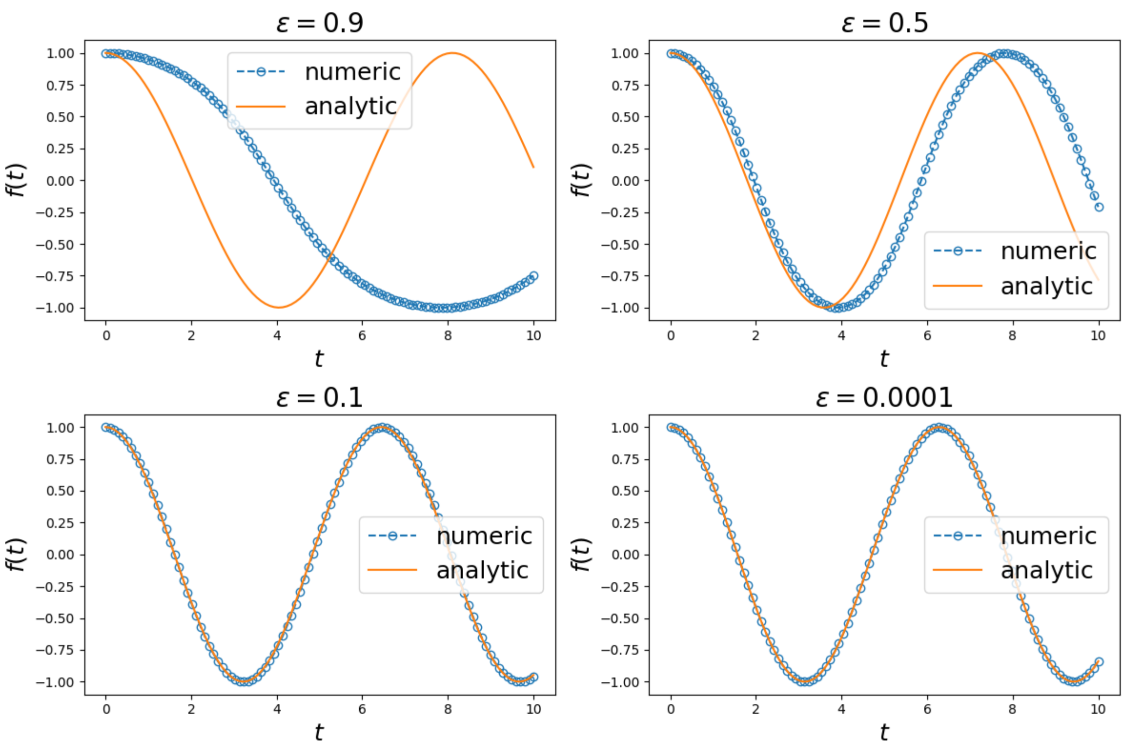

Figure 3.9: Comparison of the numeric/exact solution (blue) and first order solution (3.112) (orange) to the differential equation $\frac{d^2f}{dt^2} = -f + \epsilon f\left[1 - 2\left(\frac{df}{dt}\right)^2\right]$ with $f(0) = 1$, $df/dt(0) = 0$ for different values of ϵ. We see that the solutions converge as ϵ approaches 0.

should be a 'stretched' or 'squeezed' version of t, to account for the change of oscillation frequency. That being said, we then propose the change of variables

$$t = \tau + \epsilon\omega_1(\tau) + \epsilon^2\omega_2(\tau) + ... \tag{3.114}$$

and *substitute this into the non-uniform solution already obtained*:

$$f(\tau) = \cos(\tau + \epsilon\omega_1(\tau) + ...) + \frac{\epsilon}{8}\left(\frac{1}{4}\cos(3\tau + 3\epsilon\omega_1(\tau) + ...)\right.$$
$$\left.-\frac{1}{4}\cos(\tau + \epsilon\omega_1(\tau) + ...) - 3(\tau + \epsilon\omega_1(\tau) + ...)\sin(\tau + \epsilon\omega_1(\tau) + ...)\right) + O(\epsilon^2) \tag{3.115}$$

Let us now Taylor-expand this in ϵ, noting that

$$\cos(x + \epsilon) \simeq \cos(x) - \epsilon\sin(x) - \frac{\epsilon^2}{2}\cos(x) + ... \tag{3.116}$$

We get, keeping only terms up to $O(\epsilon)$ and neglecting terms $O(\epsilon^2)$:

$$f(\tau) = \cos(\tau) - \epsilon\omega_1(\tau)\sin(\tau) + \frac{\epsilon}{8}\left(\frac{1}{4}\cos(3\tau) - \frac{1}{4}\cos(\tau) - 3\tau\sin(\tau)\right) + O(\epsilon^2) \tag{3.117}$$

We now want to pick just the right $\omega_1(\tau)$ to 'cancel out' the non-uniform term. We see that we need

$$\omega_1(\tau) = -\frac{3}{8}\tau \tag{3.118}$$

so the correct *renormalization* of the independent variable is

$$t = \tau - \frac{3}{8}\epsilon\tau + O(\epsilon^2) \tag{3.119}$$

or equivalently

$$\tau = \frac{t}{1 - \frac{3}{8}\epsilon} + O(\epsilon^2) = \left(1 + \frac{3}{8}\epsilon\right) t + O(\epsilon^2) \quad (3.120)$$

With that, the uniform expansion to the solution of the Duffing oscillator at $O(\epsilon)$ is

$$f(\tau) = \cos(\tau) + O(\epsilon) \rightarrow f(t) = \cos\left[\left(1 + \frac{3}{8}\epsilon\right) t\right] + O(\epsilon) \quad (3.121)$$

This recovers exactly the solution obtained using the Linsted-Poincaré method.

Note that:

- By contrast with the Linsted-Poincaré technique, this time we wrote $t(\tau)$ rather that $\tau(t)$, which makes it easy to substitute into the existing solution. But that an extra step is needed at the end to invert this function to get $\tau(t)$ – this will always be the case, but sometimes the inversion may not be easy or even possible.
- Also, this time we have more flexibility because the ω_n can be arbitrary functions of τ. As a result this technique can also be applied more general classes of problems (see, e.g. Bush textbook Chapter 3)
- As for the Linsted-Poincaré technique, we started with a two-term non-uniform expansion for $f(t)$. We then obtained a two-term formula for $t(\tau)$, and finally obtained a 1-term uniform expansion for $f(t)$. In other words, we always have to go to 1 order higher in the 'naive' expansion, to find the correct rescaling of the independent variable that provides a uniform expansion at the desired order.

The Linsted-Poincaré and/or renormalization methods work very well on the problems for which they were intended. However, they do fail spectacularly when applied to problems where the nonlinearities have *other* effects on the solution, as in the example below.

Example 2: Consider the third oscillator of the list of examples provided earlier.

- Find the non-uniform expansion of the solution to $O(\epsilon^2)$
- Assume that $t = \tau + \epsilon\omega_1(\tau) + O(\epsilon^2)$ and choose ω_1 so as to cancel out the secular terms
- What happens when you try to solve for τ?

Solution: We want to solve

$$\frac{d^2 f}{dt^2} = -f + \epsilon(1 - f^2)\frac{df}{dt} \quad (3.122)$$
$$f(0) = 1 \quad (3.123)$$
$$\frac{df}{dt}(0) = 0 \quad (3.124)$$

So, let's say $f(t) = f_0(t) + \epsilon f_1(t) + O(\epsilon^2)$. Substituting this into the equation, we obtain:

$$f_0'' + \epsilon f_1'' + O(\epsilon^2) = -f_0 - \epsilon f_1 + \epsilon(1 - f_0^2 - \epsilon 2 f_0 f_1 + O(\epsilon^2))[f_0' + \epsilon f_1' + O(\epsilon^2)] \quad (3.125)$$

which, order by order, gives

$$f_0'' = -f_0 \quad (3.126)$$
$$f_1'' = -f_1 + f_0' - f_0^2 f_0' \quad (3.127)$$

The function f_0 satisfies :

$$f_0'' = -f_0 \quad (3.128)$$
$$f_0(0) = 1 \quad (3.129)$$
$$f'(0) = 0 \quad (3.130)$$

So, the solution is

$$f_0(t) = \cos(t) \tag{3.131}$$
$$\tag{3.132}$$

Substituting this into the equation for f_1 :

$$f_1'' = -f_1 - \sin(t) + \cos^2(t)\sin(t) \tag{3.133}$$
$$f_1(0) = 0 \tag{3.134}$$
$$f_1'(0) = 0 \tag{3.135}$$

Using Wolfram Alpha we have

$$\cos^2(t)\sin(t) = \frac{\sin(t)}{4} + \frac{\sin(3t)}{4} \tag{3.136}$$

So

$$f_1''(t) = -f_1(t) - \frac{3\sin(t)}{4} + \frac{\sin(3t)}{4} \tag{3.137}$$
$$f_1(0) = 0 \tag{3.138}$$
$$f_1'(0) = 0 \tag{3.139}$$

The homogeneous solution, not counting boundary conditions, is clearly

$$c_1 \sin(t) + c_2 \cos(t) \tag{3.140}$$

To solve for particular, we propose the ansatz $f(t) = At\cos(t) + Bt\sin(t) + C\cos(3t) + D\sin(3t)$ The solution is (cf. Wolfram Alpha):

$$f_1(t) = \frac{3}{8}t\cos(t) - \frac{1}{32}\sin(3t) + c_1\sin(t) + c_2\cos(t) \tag{3.141}$$

To satisfy the boundary conditions, we need $c_2 = 0$ (so $f_1(0) = 0$) and

$$\frac{3}{8} - \frac{3}{32} + c_1 = 0 \to c_1 = -\frac{9}{32} \tag{3.142}$$

So finally, we find the non-uniform expansion for $f(t)$ up to $O(\epsilon)$ to be

$$f(t) = \cos(t) + \epsilon\left[\frac{3}{8}t\cos(t) - \frac{1}{32}\sin(3t) - \frac{9}{32}\sin(t)\right] \tag{3.143}$$

Now, let's apply renormalization:

$$t = \tau + \epsilon\omega_1(t) + O(\epsilon^2) \tag{3.144}$$
$$\to f(\tau) = \cos(\tau + \epsilon\omega_1(\tau)) + \frac{3\epsilon}{8}(\tau + \epsilon\omega_1(\tau))\cos(\tau + \epsilon\omega_1(\tau)) \tag{3.145}$$
$$-\frac{\epsilon}{32}\sin(3(\tau + \epsilon\omega_1(\tau))) - \frac{9\epsilon}{32}\sin(\tau + \epsilon\omega_1(\tau)) \tag{3.146}$$

Expanding this in orders of ϵ, we obtain

$$f(\tau) = \cos(\tau) - \epsilon\omega_1(\tau)\sin(\tau) + \frac{3\epsilon}{8}\tau\cos(\tau) - \frac{\epsilon}{32}\sin(3\tau) - \frac{9\epsilon}{32}\sin(\tau) + O(\epsilon^2) \tag{3.147}$$

We see that this time the method doesn't work: if we wanted to cancel the secular $\tau\cos(\tau)$ term with the $\omega_1(\tau)\sin(\tau)$ term, we would need to choose $\omega_1(\tau) = (3\tau/8)/\tan(\tau)$. Then, the problem is that we cannot uniquely recover t from τ – the transformation is not invertible. This is a typical symptom of problems for which the renormalization method fails.

3.4 The method of multiple scales

Lecture edited by Sean, Jeremy and Alyn.

At the end of the last section, we saw an example where rescaling the independent variable did not work at all. In hindsight, this was expected, based on the nature of the solution: we had seen that in this example the nonlinearity doesn't just affect the period of the oscillator, but also causes its amplitude to change on a slow timescale. When that is the case, a different method often yields excellent results, and that method is called the *method of multiple scales*.

3.4.1 Preliminary detour

Before we dive into the method, let's first study the function

$$g(t) = e^{-\epsilon t}\sin(t) \tag{3.148}$$

which has the property of oscillating on an $O(1)$ timescale, with an amplitude that decays on an $O(\epsilon^{-1}) \gg 1$ timescale. Suppose we want to expand it as a Taylor series in ϵ, then we get

Solution:

$$e^{-\epsilon t}\sin t = \sin t - t\epsilon \sin t + \frac{1}{2}t^2\epsilon^2\sin t - ...$$

We see that this expansion is non-uniform, and is a bad approximation to the true function for large t.

However, suppose we now define two new variables, recognizing the fact that this function evolves on two vastly different timescales: a 'fast' timescale t_f and a 'slow' timescale t_s, where

$$t_f = t, t_s = \epsilon t \tag{3.149}$$

It is easy to see that when t changes by $O(1)$, t_f changes by $O(1)$ too but t_s only changes a tiny bit - by $O(\epsilon)$, making this indeed the slow time.

With these definitions, we can technically rewrite g as a function of two variables:

$$g(t) = g(t_f, t_s) = e^{t_s}\sin(t_f) \tag{3.150}$$

In this expression, furthermore, ϵ has disappeared, so no need to do any expansion.

Let's now calculate the derivative of g with respect to time. If we do it directly, we get

$$\frac{dg}{dt} = -\epsilon e^{-\epsilon t}\sin(t) + e^{-\epsilon t}\cos(t) \tag{3.151}$$

On the other hand, if we do it from the multiple timescale expression, we get

$$\begin{aligned}\frac{dg}{dt} &= \frac{\partial g}{\partial t_f}\frac{dt_f}{dt} + \frac{\partial g}{\partial t_s}\frac{dt_s}{dt} \\ &= \frac{dg}{dt_s}\cdot\epsilon + \frac{dg}{dt_f}\cdot 1 \\ &= \left(\frac{\partial}{\partial t_s} + \epsilon\frac{\partial}{\partial t_f}\right)g \\ &= e^{-t_s}\cos(t_f) - \epsilon e^{-t_s}\sin(t_f) \quad \checkmark\end{aligned}$$

Clearly, we recover the same expression. But we also see that the term containing the derivative with respect to t_s is one order of ϵ *smaller* than the term containing the derivative with respect to t_f. This is something we may be able to leverage in the context of asymptotic analysis. Let's now see how to do this on a particular simple example.

3.4.2 The weakly damped linear oscillator

Let's start with a simple linear problem, which has an easy analytical solution (that we will look for later):

$$\frac{d^2 f}{dt^2} = -f - \epsilon\frac{df}{dt} \tag{3.152}$$

$$f(0) = 1, \frac{df}{dt}(0) = 0 \tag{3.153}$$

The numerical solution to this problem suggests that the solution oscillates on an $O(1)$ timescale, and decays on a much longer timescale. Let's postulate that this longer timescale is $O(\epsilon^{-1})$, as in the previous section, and define

$$t_f = t, t_s = \epsilon t \tag{3.154}$$

as before, and let $f(t) = f(t_f, t_s)$. The first and second derivatives are

$$\frac{df}{dt} = \frac{\partial f}{\partial t_f} + \epsilon \frac{\partial f}{\partial t_s} \tag{3.155}$$

$$\frac{d^2 f}{dt^2} = \left(\frac{\partial}{\partial t_f} + \epsilon \frac{\partial}{\partial t_s}\right)^2 f = \frac{\partial^2 f}{\partial t_f^2} + 2\epsilon \frac{\partial^2 f}{\partial t_f \partial t_s} + \epsilon^2 \frac{\partial^2 f}{\partial t_s^2} \tag{3.156}$$

We *also* postulate as usual that f can be expanded as an asymptotic sequence in ϵ:

$$f(t_f, t_s) = f_0(t_f, t_s) + \epsilon f_1(t_f, t_s) + \epsilon^2 f_2(t_f, t_s) + ... \tag{3.157}$$

Substituting all of this into the governing equation, we get

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t_f^2} + 2\epsilon \frac{\partial^2}{\partial t_f \partial t_s} + \epsilon^2 \frac{\partial^2}{\partial t_s^2}\right)(f_0 + \epsilon f_1 + \epsilon^2 f_2 + ...) \\ &= -(f_0 + \epsilon f_1 + \epsilon^2 f_2 + ...) - \epsilon \left(\frac{\partial}{\partial t_f} + \epsilon \frac{\partial}{\partial t_s}\right)(f_0 + \epsilon f_1 + \epsilon^2 f_2 + ...) \end{aligned} \tag{3.158}$$

Furthermore, noting that when $t = 0$, both $t_f = 0$ and $t_s = 0$, the initial conditions become

$$f_0(0,0) + \epsilon f_1(0,0) + \epsilon^2 f_2(0,0) + ... = 1 \tag{3.159}$$

$$\left(\frac{\partial}{\partial t_f} + \epsilon \frac{\partial}{\partial t_s}\right)(f_0 + \epsilon f_1 + \epsilon^2 f_2 + ...) = 0 \text{ at } t_f = t_s = 0 \tag{3.160}$$

Let's now look at these equations order by order in ϵ. The lowest possible order terms are $O(1)$ (no ϵ). At this order we have

$$\frac{\partial^2 f_0}{\partial t_f^2} = -f_0 \tag{3.161}$$

$$f_0(0,0) = 1, \frac{\partial f_0}{\partial t_f}(0,0) = 0 \tag{3.162}$$

We know that the solution should be a linear combination of $\cos(t_f)$ and $\sin(t_f)$. However, because the equation for f_0 is now a PDE (rather than an ODE), the coefficients are allowed to be arbitrary function of t_s:

$$f_0(t_f, t_s) = A_0(t_s)\cos(t_f) + B_0(t_s)\sin(t_f) \tag{3.163}$$

(check this!) The initial conditions then imply that

$$A_0(0) = 1, B_0(0) = 0 \tag{3.164}$$

We see that they fix the values of these functions at $t_s = 0$, but we are still in the dark as to what $A_0(t_s)$ and $B_0(t_s)$ are for $t_s > 0$. This is obtained from the next order in the expansion.

Let's now look at the equation at $O(\epsilon)$. We have

$$\frac{\partial^2 f_1}{\partial t_f^2} + 2\frac{\partial^2 f_0}{\partial t_f \partial t_s} = -f_1 - \frac{\partial f_0}{\partial t_f} \tag{3.165}$$

Using what we know of f_0, this can be rewritten as

$$\begin{aligned} \frac{\partial^2 f_1}{\partial t_f^2} + f_1 &= -\left[-A_0(t_s)\sin(t_f) + B_0(t_s)\cos(t_f)\right] \\ &\quad -2\left[-\frac{dA_0}{dt_s}\sin(t_f) + \frac{dB_0}{dt_s}\cos(t_f)\right] \end{aligned} \tag{3.166}$$

$$= \left[A_0(t_s) + 2\frac{dA_0}{dt_s}\right]\sin(t_f) - \left[B_0(t_s) + 2\frac{dB_0}{dt_s}\right]\cos(t_f) \tag{3.167}$$

The solution of this non-homogeneous equation for f_1 is the sum of a general solution plus a particular solution, i.e. of the form

$$f_1(t_f, t_s) = A_1(t_s)\cos(t_f) + B_1(t_s)\sin t_f + \text{ part. sol.} \tag{3.168}$$

with the particular solution proportional to $t_f \cos(t_f)$ and $t_f \sin(t_f)$. These terms are unbounded, and would be a source of non-uniformity in a solution that (numerically) looks like it is otherwise bounded. So we would like to avoid them – and doing so requires that the right-hand side (3.167) be zero, which can be satisfied provided

$$A_0(t_s) + 2\frac{dA_0}{dt_s} = 0 \tag{3.169}$$

$$B_0(t_s) + 2\frac{dB_0}{dt_s} = 0 \tag{3.170}$$

These are the equations for $A_0(t_s)$ and $B_0(t_s)$ that we were looking for, and are sometimes called the *compatibility condition* and sometimes called the *solvability condition*.

We can easily solve the compatibility solution, subject to initial conditions $A_0(0) = 1, B_0(0) = 0$: we find

$$A_0(t_s) = e^{-t_s/2}, B_0(t_s) = 0 \tag{3.171}$$

So finally, we find that the solution for $f(t_f, t_s)$ is

$$f(t_f, t_s) = f_0(t_f, t_s) + O(\epsilon) = e^{-t_s/2}\cos(t_f) + O(\epsilon) \tag{3.172}$$

so

$$f(t) = e^{-\epsilon t/2}\cos(t) + O(\epsilon) \tag{3.173}$$

We can (somewhat easily) check that this recovers the lowest-order expansion of the exact solution for $f(t)$ in small ϵ:

Solution:

$$e^{-\epsilon t/2}\cos(t) = \cos t - \frac{t}{2}\epsilon\cos t + \frac{1}{8}t^2\epsilon^2\cos t + O(\epsilon^3)$$

3.4.3 The van der Pol oscillator

The van der Pol oscillator is one of the most famous examples of a nonlinear oscillator that has a stable limit cycle. It is used to model many systems in electrical engineering, biological systems, chemistry, etc. It is modeled by the equation

$$\frac{d^2 f}{dt^2} + f = \epsilon(1 - f^2)\frac{df}{dt} \tag{3.174}$$

and we will assume $f(0) = h$, $df/dt(0) = 0$.

Physically, we see that the nonlinear term acts to amplify f if $f < 1$ and acts to damp f if $f > 1$ – so it is not really surprising that we should end up with a limit cycle, which is what we found a few lectures ago.

Let's now study the problem using multiscale analysis – this seems to be a good candidate, because the solution oscillate with a frequency of $O(1)$ but the amplitude varies over long timescales.

- Let $t_f = t$, $t_s = \epsilon t$ and $f = f_0 + \epsilon f_1 + ...$ Find the 0th and 1st order equations and associated initial conditions.
- Solve the 0th order problem, and show that the solution can be written as

$$f_0(t_f, t_s) = A_0(t_s)e^{it_f} + A_0^*(t_s)e^{-it_f} \tag{3.175}$$

which will turn out to be easier to deal with than if we had written it using sines and cosines.

- Substitute this solution into the 1st order equation, collect terms in e^{it_f} and e^{-it_f}, and show that the compatibility condition is
$$\frac{dA_0}{dt_s} = \frac{1}{2}A_0(1-|A_0|^2) \tag{3.176}$$
- Use the exponential notation $A_0(t_s) = |A_0(t_s)|e^{i\theta_0(t_s)}$ to obtain two equations for $|A_0|$ and θ_0, then solve them to obtain the time-dependence of the amplitude and phase of the 0th order oscillator.

Solution: We begin by computing the derivatives using the multiscale assumption:

$$\frac{d}{dt} = \frac{dt_f}{dt}\frac{\partial}{\partial t_f} + \frac{dt_s}{dt}\frac{\partial}{\partial t_s} = \frac{\partial}{\partial t_f} + \epsilon\frac{\partial}{\partial t_s} \tag{3.177}$$
$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t_f} + \epsilon\frac{\partial}{\partial t_s}\right)^2 = \frac{\partial^2}{\partial t_f^2} + 2\epsilon\frac{\partial^2}{\partial t_f \partial t_s} + \epsilon^2\frac{\partial^2}{\partial t_s^2} \tag{3.178}$$

If we also assume that $f = f_0 + \epsilon f_1 + ...$ then the governing equation is:

$$\left(\epsilon\frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}\right)^2 (f_0 + \epsilon f_1 + \dots) + (f_0 + \epsilon f_1 + \dots) = \tag{3.179}$$
$$\epsilon\left(\epsilon\frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}\right)(f_0 + \epsilon f_1 + \dots)(1 - (f_0 + \epsilon f_1 + \dots)^2) \tag{3.180}$$

Observing the $\mathcal{O}(\epsilon^0)$ terms:

$$\frac{\partial^2 f_0}{\partial t_f^2} + f_0 = 0 \tag{3.181}$$

Given this differential equation, we know that the solution is some linear combination of sines and cosines with non-constant coefficients dependent on t_s

$$f_0(t_f, t_s) = \alpha(t_s)\cos(t_f) + \beta(t_s)\sin(t_f)$$

However, cross-terms in the $\mathcal{O}(\epsilon^1)$ terms will mean breaking f_0 into sines and cosines in this scenario will make the algebra along the way extremely unwieldly; we can instead change the sines and cosines to be (via their respective Euler's identities):

$$\begin{aligned} f_0(t_f, t_s) &= \alpha(t_s)(\frac{e^{it_f} + e^{-it_f}}{2}) + \beta(t_s)(\frac{e^{it_f} - e^{-it_f}}{2i}) \\ &= \alpha(t_s)(\frac{e^{it_f} + e^{-it_f}}{2}) - i\beta(t_s)(\frac{e^{it_f} - e^{-it_f}}{2}) \\ &= \frac{1}{2}\left(\alpha(t_s) - i\beta(t_s)\right)e^{it_f} + \frac{1}{2}\left(\alpha(t_s) + i\beta(t_s)\right)e^{-it_f} \end{aligned}$$

We can define new non-constant coefficients A and B in terms of the earlier non-constant coefficients α and β above. However, note that each complex coefficient is just the other's conjugate. So we simply define $A(t_s) := \frac{1}{2}\left(\alpha(t_s) - i\beta(t_s)\right) \implies A^*(t_s) = \frac{1}{2}\left(\alpha(t_s) + i\beta(t_s)\right)$.
Thus we know the form of f_0 to be

$$f_0(t_f, t_s) = A(t_s)e^{it_f} + A^*(t_s)e^{-it_f}$$

Observing $O(\epsilon^1)$ terms:

$$\frac{\partial^2 f_1}{\partial t_f^2} + 2\frac{\partial^2 f_0}{\partial t_s \partial t_f} + f_1 = \frac{\partial f_0}{\partial t_f}(1 - f_0^2) \tag{3.182}$$
$$\frac{\partial^2 f_1}{\partial t_f^2} + f_1 = (1 - f_0^2)\frac{\partial f_0}{\partial t_f} - 2\frac{\partial^2 f_0}{\partial t_s \partial t_f} \tag{3.183}$$
$$= \left(1 - \left(Ae^{it_f} + A^*e^{-it_f}\right)^2\right)\left(iAe^{it_f} - iA^*e^{-it_f}\right) - 2\left(i\frac{dA}{dt_s}e^{it_f} - i\frac{dA^*}{dt_s}e^{-it_f}\right) \tag{3.184}$$

The right-hand side of (3.184) comes down to:

- terms that will generate particular solutions that do not grow with time e.g. terms in $e^{i2t}, e^{i3t}, e^{i5t}$...
- terms that will lead to particular solutions that grow with time: i.e. terms that are proportional to e^{it_f} or e^{-it_f}

Analyzing terms in proportion to e^{it_f}:

$$iA + iA^2A^* - 2iA^2A^* - 2i\frac{dA}{dt_s}$$

and similarly for terms in e^{-it_f} (which turn out to be the complex conjugate of the ones we just wrote). Setting this expression to zero to 'kill' the source of secular growth yields the ODE:

$$\frac{dA}{dt_s} = \frac{A}{2} - \frac{A^2A^*}{2}$$

Recall that $AA^* = |A|^2$ so

$$\frac{dA}{dt_s} = \frac{A}{2}(1 - |A|^2)$$

Look for solutions of the form

$$A = |A(t_s)|e^{i\theta(t_s)}$$

Then

$$e^{i\theta(t_s)}\frac{d|A|}{dt_s} + i|A|\frac{d\theta(t_s)}{dt_s}e^{i\theta(t_s)} = \frac{1}{2}|A|e^{i\theta(t_s)}(1 - |A|^2) \tag{3.185}$$

We now divide by $e^{i\theta}$ and analyze the real and imaginary parts of this equation individually:

- Imaginary:

  $$|A|\frac{d\theta(t_s)}{dt_s} = 0$$

  so $\theta(t_s)$ is constant.
- Real Part:

  $$\frac{d|A|}{dt_s} = \frac{|A|}{2}(1 - |A|^2)$$

  which is separable.

We solve it using:

$$\frac{d|A|}{|A|(1 - |A|^2)} = \frac{dt_s}{2}$$

Using integral tables we find the solution to be:

$$\ln\sqrt{\frac{|A|^2}{|1 - |A|^2|}} = \frac{t_s}{2} + C$$

Applying the initial conditions we know that when $t_s = 0$ we need $f_0(0, 0) = h$ thus

$$A(0)e^0 + A^*(0)e^0 = h \Rightarrow 2Re(A) = h$$

From the second condition initial condition, $\frac{\partial f_0}{\partial t_f}(0) = 0$ and thus

$$iA(0)e^0 - iA^*(0)e^0 = 0 \Rightarrow A(0) = A^*(0)$$

and thus $A(0)$ is real, and combining that with the first condition we obtain that at time $t_s = 0$, $A(0) = \frac{h}{2}$. Plugging these into the equation we get

$$\ln\sqrt{\frac{\frac{h^2}{4}}{|1 - \frac{h^2}{4}|}} = 0 + C \rightarrow C = \ln\sqrt{\frac{h^2}{|4 - h^2|}}$$

Therefore the full equation is

$$\frac{|A|^2}{|1 - |A|^2|} = e^{t_s}\frac{h^2}{|4 - h^2|}$$

This leaves two cases for A

- if $|A| < 1$ (so $|h| < 2$) the equation becomes

$$\frac{|A|^2}{1-|A|^2} = e^{t_s}\frac{h^2}{4-h^2}$$

$$|A|^2 = \frac{e^{t_s}\frac{h^2}{4-h^2}}{1+e^{t_s}\frac{h^2}{4-h^2}} = \frac{1}{1+e^{-t_s}\frac{4-h^2}{h^2}}$$

  which is always less than one

- if $|A| > 1$ (so $|h| > 2$)

$$\frac{|A|^2}{|A|^2-1} = e^{t_s}\frac{h^2}{h^2-4}$$

$$|A|^2 = \frac{1}{1-e^{-t_s}\frac{h^2-4}{h^2}}$$

  which is always greater than one

Thus finally we come to the conclusion that

$$\begin{aligned} f_o(t_s,t_f) &= A(t_s)e^{it_f} + A^*(t_s)e^{-it_f} \\ &= A(t_s)(e^{it_f} + e^{-it_f}) \\ &= \frac{2\cos t_f}{\sqrt{1+e^{-t_s}\frac{4-h^2}{h^2}}} \end{aligned}$$

thus by replacing t_s and t_f we recover the first order multi-scale solution of the Van der Pol oscillator

$$f_0(t) = \frac{2\cos t}{\sqrt{1+e^{-\epsilon t}\frac{4-h^2}{h^2}}}$$

This shows that as $t \to \infty, f_0 \to 2\cos t$

Note:

- In these types of problems, the exponential notation for sines and cosines, and for complex numbers, will almost always lead more straightforwardly to the solution. To convince yourself of that, try to redo the van der Pol problem using sines and cosines instead for the f_0 solution.

- In both examples studied so far, we assumed a scaling for t_f and t_s and an asymptotic expansion for f, and we obtained a meaningful solution. That is *because* we had made the correct choices for the new variables and for f.

- In this course, you will always be provided with the right expansion and proposed variables. In real life, should you ever have to solve an asymptotic problem, you will be the one choosing them.

- It is not always the case that $t_f = t$, $t_s = \epsilon t$ is the right choice for the new variables, and it is not always the case that an asymptotic series in ϵ is the right choice for f! Much of the *art* of asymptotic analysis is to *find* the right variables and the right expansions. This is where taking a peek at the numerical solution can really help make informed decisions.

3.5 WKB theory

Lecture edited by Charlie and Julian

WKB theory (where WKB stands for Wentzel, Kramers and Brillouin), is an extension of the multiscale method that works more generally when the basic multiscale method described in the previous lecture fails.

3.5.1 The failure of the multiscale method

Indeed, the multiscale method does not always work. In fact, it can fail rather spectacularly even in some simple problems where one may naively think that it should work. Consider for instance

$$\frac{d^2f}{dt^2} = -\frac{w^2(t)}{\epsilon^2}f(t) \tag{3.186}$$

where $\epsilon \ll 1$, and $w^2(t) > 0$ and $w(t) = O(1)$ for all t. This is essentially the equation for an oscillator with a frequency $\omega(t) = w(t)/\epsilon$ so the oscillation period is very short, $O(\epsilon)$, but this frequency varies 'slowly' on an $O(1)$ timescale. This looks like an ideal candidate for the method of multiple scales, so let's see what happens when we try to use it.

We define the slow and fast timescales as

$$t_s = t, \quad t_f = \frac{t}{\epsilon} \tag{3.187}$$

(this time, the slow timescale is $O(1)$ and the fast timescale is $O(\epsilon)$), and look for solutions $f(t_s, t_f) = f_0(t_s, t_f) + \epsilon f_1(t_s, t_f) + ...$

This time

$$\frac{d}{dt} = \frac{\partial}{\partial t_s} + \frac{1}{\epsilon}\frac{\partial}{\partial t_f} \tag{3.188}$$

and so

$$\frac{d^2f}{dt^2} = \left(\frac{\partial^2}{\partial t_s^2} + \frac{2}{\epsilon}\frac{\partial^2}{\partial t_s \partial t_f} + \frac{1}{\epsilon^2}\frac{\partial^2}{\partial t_f^2}\right)(f_0 + \epsilon f_1 + ...) = -\frac{w^2(t_s)}{\epsilon^2}(f_0 + \epsilon f_1 + ...) \tag{3.189}$$

At the lowest order (which is $O(\epsilon^{-2})$), we now have

$$\frac{\partial^2 f_0}{\partial t_f^2} = -w^2(t_s)f_0 \tag{3.190}$$

This is *almost* as we had in previous examples, except that this time the lowest-order equation in the fast variable contains slowly varying coefficients. As we shall see, that will turn out to be highly problematic. But for now, let's proceed as we normally would. We would write

$$f_0(t_s, t_f) = a(t_s)\cos(w(t_s)t_f) + b(t_s)\sin(w(t_s)t_f) \tag{3.191}$$

and hope to obtain equations for the slowly varying amplitudes at the next order in ϵ.

At the next order (which is $(O(\epsilon^{-1}))$, we have

$$\frac{\partial^2 f_1}{\partial t_f^2} + w^2(t_s)f_1 = -2\frac{\partial^2 f_0}{\partial t_s \partial t_f} \tag{3.192}$$

and we hope to get the amplitude equation by setting the 'secular' terms to 0. Let's compute the right-hand side.

$$\begin{aligned}
-2\frac{\partial^2 f_0}{\partial t_s \partial t_f} &= -2\frac{\partial}{\partial t_s}\left[-a(t_s)w(t_s)\sin(w(t_s)t_f) + b(t_s)w(t_s)\cos(w(t_s)t_f)\right] \\
&= 2\frac{\partial}{\partial t_s}(a(t_s)w(t_s))\sin(w(t_s)t_f) + 2a(t_s)w(t_s)t_f\frac{\partial w}{\partial t_s}\cos(w(t_s)t_f) \\
&\quad -2\frac{\partial}{\partial t_s}(b(t_s)w(t_s))\cos(w(t_s)t_f) + 2b(t_s)w(t_s)t_f\frac{\partial w}{\partial t_s}\sin(w(t_s)t_f)
\end{aligned}$$

We see that there is no way of choosing $a(t_s)$ and $b(t_s)$ to eliminate all of these terms (unless they are both 0, which is not the solution we are looking for). This shows that the basic multiple scale assumption fails here.

In hindsight, this is not entirely surprising. Because the frequency of the oscillation is proportional to $w(t)$, it evolves with t, and so the correct 'fast' timescale also varies with time (rather than being just proportional to t/ϵ everywhere). The WKB method, presented below, therefore generalizes the multiscale method to allow for a *nonlinear* relationship between the fast and regular timescales. In some sense, it is not unlike what we did in the renormalization method earlier on, but this time, we apply it when we have 2 timescales.

3.5.2 The WKB assumption, and solution

A simple way to remedy the problem is to allow t_f to be a nonlinear function of t, and here more specifically let's define

$$t_f = \frac{g(t)}{\epsilon}, \quad t_s = t \tag{3.193}$$

where we will choose $g(t)$ wisely to get the method to work. With this choice, we have

$$\frac{d}{dt} = \frac{\partial}{\partial t_s} + \frac{g'(t_s)}{\epsilon}\frac{\partial}{\partial t_f} \tag{3.194}$$

so

$$\begin{aligned} \frac{d^2 f}{dt^2} &= \left[\frac{\partial^2}{\partial t_s^2} + \frac{g''(t_s)}{\epsilon}\frac{\partial}{\partial t_f} + 2\frac{g'(t_s)}{\epsilon}\frac{\partial^2}{\partial t_f \partial t_s} + \frac{[g'(t_s)]^2}{\epsilon^2}\frac{\partial^2}{\partial t_f^2}\right](f_0 + \epsilon f_1 + ...) \\ &= -\frac{w^2(t_s)}{\epsilon^2}(f_0 + \epsilon f_1 + ...) \end{aligned} \tag{3.195}$$

This time, at the lowest order we get

$$[g'(t_s)]^2 \frac{\partial^2 f_0}{\partial t_f^2} = -w^2(t_s) f_0 \tag{3.196}$$

and we can choose $g' = |w|$, or in other words $g = \int |w|$ to ensure that

$$f_0(t_s, t_f) = a(t_s)\cos(t_f) + b(t_s)\sin(t_f) \tag{3.197}$$

We find the coefficients $a(t_s)$ and $b(t_s)$ at the next order:

$$[g'(t_s)]^2 \frac{\partial^2 f_1}{\partial t_f^2} + w^2(t_s) f_1 = -g''(t_s)\frac{\partial f_0}{\partial t_f} - 2g'(t_s)\frac{\partial^2 f_0}{\partial t_f \partial t_s} \tag{3.198}$$

Substituting what we know of f_0 and g, we get

$$[g'(t_s)]^2 \left(\frac{\partial^2 f_1}{\partial t_f^2} + f_1\right) = -g''(t_s)\left(-a(t_s)\sin(t_f) + b(t_s)\cos(t_f)\right) \tag{3.199}$$

$$-2g'(t_s)\left(-a'(t_s)\sin(t_f) + b'(t_s)\cos(t_f)\right) \tag{3.200}$$

The 'secular' terms on the right-hand side can be eliminated provided

$$ag'' + 2a'g' = 0 \text{ and } bg'' + 2b'g' = 0 \tag{3.201}$$

The a equation can be solved to give:

$$\frac{a'}{a} = -\frac{1}{2}\frac{g''}{g'} \rightarrow a(t_s) = A\exp(-\frac{1}{2}\ln(g'(t_s))) = \frac{A}{\sqrt{g'(t_s)}} = \frac{A}{\sqrt{|w(t_s)|}} \tag{3.202}$$

(where A is some integration constant) and similarly for $b(t_s)$ (since it's exactly the same equation).

Putting everything together, we now get that at lowest order, the WKB solution is

$$f_0(t_s, t_f) = \frac{1}{\sqrt{|w(t_s)|}}\left(A\cos(t_f) + B\sin(t_f)\right) \tag{3.203}$$

where A and B are integration constants. Substituting back $t_s = t$, and

$$t_f = \frac{g(t)}{\epsilon} = \frac{1}{\epsilon}\int_0^t |w(t')|dt' \tag{3.204}$$

(which ensures that $t_f = 0$ when $t = 0$) we finally get the well-known WKB formula for the solution of ODEs of the form (3.186), namely

$$f(t) = \frac{1}{\sqrt{|w(t)|}}\left[A\cos(\epsilon^{-1}\int_0^t |w(t')|dt') + B\sin(\epsilon^{-1}\int_0^t |w(t')|dt')\right] \tag{3.205}$$

Example: Find the WKB solution of

$$\frac{d^2 f}{dt^2} = -\frac{(t^2+1)^2}{\epsilon^2} f(t) \tag{3.206}$$

subject to $f(0) = 1$, $df/dt(0) = 0$.

Solution: Comparing (3.186) with (3.206) we see that we have $w(t) = t^2 + 1$ from . We know our solution has the form (3.205), and just plug in our new $w(t)$. This looks like:

$$f(t) = \frac{1}{\sqrt{|w(t)|}}\left[A\cos(\epsilon^{-1}\int_0^t |w(t)|dt') + B\sin(\epsilon^{-1}\int_0^t |w(t)|)dt')\right] \tag{3.207}$$

$$= \frac{1}{\sqrt{1+t^2}}\left[A\cos(\epsilon^{-1}\int_0^t (t'^2+1)dt') + B\sin(\epsilon^{-1}\int_0^t (t'^2+1)dt')\right] \tag{3.208}$$

Now we can apply our initial conditions $f(0) = 1$ and $\frac{df}{dt}(0) = 0$

$$f(0) = A = 1 \tag{3.209}$$

$$\frac{df}{dt} = \frac{1}{\sqrt{1+t^2}}\left[-A(\frac{1}{\epsilon}(1+t^2))\sin(\epsilon^{-1}(\frac{t^3}{3}+t)) + B(\frac{1}{\epsilon}(1+t^2))\cos(\epsilon^{-1}(\frac{t^3}{3}+t))\right] + O(1) \tag{3.210}$$

$$\implies \frac{df}{dt}(0) = \frac{B}{\epsilon} = 0 \implies B = 0 \tag{3.211}$$

We assemble everything and end up with:

$$f(t) = \frac{1}{\sqrt{1+t^2}}\cos\left[\epsilon^{-1}(\frac{t^3}{3}+t)\right] \tag{3.212}$$

Note: All of the above requires that $w(t)$ be strictly positive or strictly negative everywhere in the domain considered. If $w(t)$ is 0 somewhere, then we have a so-called 'turning point' which needs to be treated differently (see Bush for detail).

3.5.3 The WKB solution for the large-eigenvalue limit of Sturm-Liouville problems

The same method can be used to derive asymptotic approximations to the large-eigenvalue limit of regular SL problems of the form

$$\frac{d}{dx}\left(p(x)\frac{df}{dx}\right) + q(x)f = -\lambda w(x)f(x) \tag{3.213}$$

where $p(x) > 0$, $w(x) > 0$, and where we assume that $\lambda \gg 1$ (the large eigenvalue limit).

To do this

- Let $\lambda = \epsilon^{-2}$
- Let $x_s = x$ and $x_f = g(x)/\epsilon$, and $f(x) = f_0(x_s, x_f) + \epsilon f_1(x_s, x_f) + ...$

- Show that the lowest order equation is

$$p(x_s)[g'(x_s)]^2 \frac{\partial^2 f_0}{\partial t_f^2} = -w(x_s) f_0 \tag{3.214}$$

  Deduce what $g(x)$ is, and find the 0th order solution to the problem. Your solution should include two unknown functions $a(x_s)$ and $b(x_s)$.

- Find the equation for f_1 at the next order, and show that eliminating the secular terms implies

$$a(x_s) = A[w(x_s)p(x_s)]^{-1/4} \tag{3.215}$$

  and similarly for $b(x_s)$. Note how and why in practice, the $q(x)$ term never matters.

- Construct the final solution, and show that it recovers the expression from Section 2.4.8.

3.6 Boundary layer theory for singular problems

In section 3.2.6, we studied various sources of non-uniformity, and saw that it can also arise when the small parameter ϵ multiplies the highest-order derivative in the equation. This section introduces boundary layer theory as a possible technique to solve these types of equations (noting that other techniques also exist).

3.6.1 What is a boundary layer and how to find them

Let us consider a two-point boundary value problem on the interval $[x_a, x_b]$, with a simple second order ODE of the form

$$\epsilon \frac{d^2 f}{dx^2} + \mathcal{L}_1(f) = 0 \tag{3.216}$$

where $\mathcal{L}_1(f)$ is a first order differential operator (which may or may not be homogeneous). The concepts that are introduced below apply to higher-order, nonlinear ODEs as well, but it is simpler to present them for second order linear ODEs. Because this equation is second-order, we need two boundary conditions, say $f(x_a) = f_a$ and $f(x_b) = f_b$.

In the limit that $\epsilon = 0$, the ODE reduces to the first order ODE

$$\mathcal{L}_1(f) = 0 \tag{3.217}$$

whose solution can *only* satisfy one of the two boundary conditions – not both! So in the strict limit $\epsilon = 0$, the ODE does *not* have any solution unless the boundary conditions are (by some stroke of luck) just right for the solution of the 1st order problem. In general, they are not.

However, as soon as $\epsilon \neq 0$, the equation becomes a second-order ODE and so ought to satisfy both boundary conditions. It therefore seems that

- the $\epsilon d^2 f/dx^2$ term is very important in helping f fit the boundary conditions;
- more specifically, that term has to become a dominant contribution to the equation somewhere in the domain, even though it contains a small ϵ. This suggests that $d^2 f/dx^2$ must become very large somewhere in the domain (in other words, the slope of f changes very rapidly)

Let's now see how this manifests itself on a few different equations and associated boundary conditions.

Examples:

- Example 1: $\epsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} = 2x + 1$ with $f(0) = 1$, $f(1) = 4$.
- Example 2: $\epsilon \frac{d^2 f}{dx^2} - \frac{df}{dx} + f = 0$ with $f(0) = 1$, $f(1) = 2$.
- Example 3: $\epsilon \frac{d^2 f}{dx^2} + x\frac{df}{dx} + xf = 0$ with $f(-1) = e$, $f(1) = 2/e$.

We can use Matlab with the intrinsic function bvp4c to plot the solutions for different values of ϵ.

Solution:

We see that

- There are indeed some regions of the solution where the slope of f changes very rapidly with x. Taking a 'peek' at the numerical solution for moderate ϵ helps find out where these regions are.
- These are attached to a boundary in examples 1 and 2 (there are called *boundary layers*, but are in the middle of the domain for example 3 (this is an example of an *internal boundary layer*).
- These regions become thinner and thinner as $\epsilon \to 0$
- For sufficiently small ϵ, the numerical solution becomes under-resolved / bvp4c crashes. An asymptotic solution that exists for all ϵ would be nice...

3.6.2 Inner and outer solutions, composite expansions

The 'trick' to finding solutions of ODEs that exhibit boundary layers is to solve them separately outside of the boundary layer (the solution is then called the *outer solution*), and inside the boundary layer (the solution is then called the *inner solution*). We then (somehow, see below) match these two solutions at the edge of the boundary layer. Let us work through a few examples to see how this works in practice. Note that in all that follows, we will only attempt to find the *lowest order* term in the asymptotic expansion of the solution in ϵ. Higher-order boundary layer solutions are more difficult to treat, and we leave them to a course dedicated to asymptotic theory (see, e.g., Bush textbook for more detail).

Example 1: We saw a few lectures ago the problem

$$\epsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} = 2x + 1 \tag{3.218}$$

with $f(0) = 1$, $f(1) = 4$.

The outer solution: to find the lowest-order term in the outer solution is quite simple: just set $\epsilon = 0$ and solve the resulting equation

$$\frac{df_{out}}{dx} = 2x + 1 \rightarrow f_{out}(x) = x^2 + x + c_{out} \tag{3.219}$$

where c_{out} is an integration constant. Since the boundary layer is attached to the $x = 0$ boundary (see above), we must require that the outer solution satisfies the $x = 1$ boundary condition: $f_{out}(x = 1) = 4$, which implies $c_{out} = 2$.

The inner solution: finding the inner solution is a little trickier, because we do not know a priori what is the *dominant balance* in the boundary layer equation. Indeed, we know that within the boundary layer, the term $\epsilon d^2 f/dx^2$ must become important, so the relevant equation is no longer $df/dx \simeq 2x + 1$, but it could be either

$$\epsilon \frac{d^2 f_{in}}{dx^2} \simeq 2x + 1 \tag{3.220}$$

or

$$\epsilon \frac{d^2 f_{in}}{dx^2} \simeq -\frac{df_{in}}{dx} \tag{3.221}$$

Which one is it?

To find out, we first *rescale* the x coordinate because we know that the solution varies rapidly over the boundary layer (this is very reminiscent of the multiscale method, only here, we only apply it in the boundary layer). We therefore let

$$s = \frac{x}{\epsilon^\alpha} \tag{3.222}$$

where $\alpha > 0$. With this definition $s = 0$ when $x = 0$ (at the boundary), and $s = O(1)$ when $x = O(\epsilon^\alpha)$ (within the boundary layer). Note how we have not yet said what α is – that is something we have to determine as part of the process. With this rescaling, we have

$$\epsilon^{1-2\alpha} \frac{d^2 f}{ds^2} + \epsilon^{-\alpha} \frac{df}{ds} = 2s\epsilon^\alpha + 1 \simeq 1 \tag{3.223}$$

because, by definition $s = O(1)$ in the boundary layer so $s\epsilon^\alpha$ is small compared with 1. Also note that the boundary condition to apply is $f(s = 0) = 1$ so $f = O(1)$ in the boundary layer.

The two possibilities above then become either

$$\epsilon^{1-2\alpha} \frac{d^2 f}{ds^2} \simeq 1 \quad \text{neglecting } \epsilon^{-\alpha} \frac{df}{ds} \tag{3.224}$$

or

$$\epsilon^{1-2\alpha} \frac{d^2 f}{ds^2} \simeq -\epsilon^{-\alpha} \frac{df}{ds} \quad \text{neglecting } 1 \tag{3.225}$$

To find out which of the two possibilities is correct, we inspect the sizes of each term, deduce what α should be, and check for self-consistency.

In the first case, $f = O(1)$ means that $\epsilon^{1-2\alpha}$ must also be $O(1)$, which can only happen when $\alpha = 1/2$. However, if that is the case the discarded term $\epsilon^{-1/2} df/ds$ ends up much larger than the terms that were kept, leading to an inconsistency.

We are therefore left with the second case, which tells us that

$$\epsilon^{1-2\alpha} = \epsilon^{-\alpha} \tag{3.226}$$

and the only way this is possible is if $\alpha = 1$. In that case, the terms that are kept are $O(\epsilon^{-1})$, which is indeed much larger than the neglected term which is $O(1)$.

Having found the dominant balance (and α) we can find the *inner* solution, which is the solution of (3.225) with $\alpha = 1$, namely $d^2 f_{in}/ds^2 = -df_{in}/ds$. Integrating once

$$\frac{df_{in}}{ds} = c_{in} e^{-s} \tag{3.227}$$

and integrating a second time

$$f_{in}(s) = -c_{in} e^{-s} + d_{in} \tag{3.228}$$

where c_{in} and d_{in} are integration constants.

We can apply the boundary condition at $s = 0$ ($f_{in}(0) = 1$) which is indeed in the inner region. This gives

$$-c_{in} + d_{in} = 1 \tag{3.229}$$

so

$$f_{in}(s) = (1 - d_{in}) e^{-s} + d_{in} \tag{3.230}$$

We still have one constant left to find, and that one can only be found by requiring that the inner and outer solutions be compatible with one another at the outer edge of the boundary layer. To do so, we require what is called *Prandtl's matching condition*, namely that

The limit of the inner solution as you go out of the boundary layer equals the limit of the outer solution as you go into the boundary layer.

For this problem here, *going out of the boundary layer* means $s \to +\infty$ while *going into the boundary layer* means $x \to 0$. So we need

$$\lim_{s\to+\infty} f_{in}(s) = d_{in} = \lim_{x\to 0} f_{out}(x) = c_{out} = 2 \tag{3.231}$$

This means that $d_{in} = 2$!

The composite expansion: The final step of the whole process is to find a solution that is valid everywhere in the domain. So far we have, remembering that $s = x/\epsilon$

$$f_{in}(x) = 2 - e^{-x/\epsilon} \tag{3.232}$$

$$f_{out}(x) = x^2 + x + 2 \tag{3.233}$$

We then construct a *composite* expansion

$$f_{comp}(x) = f_{in}(x) + f_{out}(x) - L \tag{3.234}$$

where L is the common limit of Prandtl's matching condition (which here was $L = 2$). This means that

$$f_{comp}(x) = 2 - e^{-x/\epsilon} + x^2 + x \tag{3.235}$$

We can see using the Matlab code provided how well this solution matches the numerical solution.

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Example 2:

$$\epsilon \frac{d^2 f}{dx^2} - \frac{df}{dx} + f = 0 \tag{3.236}$$

with $f(0) = 1$, $f(1) = 2$. We now apply a similar method to this example.

- Remind yourself (from the numerical solution) where the boundary layer actually is.
- Find the outer solution. Identify the integration constant by applying the appropriate boundary condition.
- Define the rescaled variable in the boundary layer as $s = (1 - x)/\epsilon^\alpha$. (Note that you could also define $s = (x - 1)/\epsilon^\alpha$ but then s would be negative everywhere).
- Write the ODE in s, and use arguments of dominant balance to explain why the inner balance is

$$\epsilon \frac{d^2 f_{in}}{ds^2} = \frac{df_{in}}{ds} \tag{3.237}$$

- Solve the inner equation and apply the relevant boundary condition in the boundary layer to eliminate one of the integration constants.
- Find the remaining integration constant by applying Prandtl's matching condition
- Construct the composite expansion, and compare it with the numerical solution (and the exact analytical solution)

Solution:

1. Boundary Layer Location: From the numerical solution, the boundary layer is near $x = 1$.
2. Outer Solution: For $\epsilon \to 0$, the term $\epsilon \frac{d^2 f}{dx^2}$ becomes negligible, so the outer equation reduces to:

$$-\frac{df_{out}}{dx} + f_{out} = 0.$$

Solving gives:

$$f_{out}(x) = C_1 e^x.$$

Using the boundary condition $f(0) = 1$, we find $C_1 = 1$, so:

$$f_{out}(x) = e^x.$$

3. Define the Rescaled Variable: To analyze the boundary layer near $x = 1$, introduce the rescaled variable:

$$s = \frac{x - 1}{\epsilon^\alpha}.$$

Then:

$$\frac{d}{dx} = \frac{1}{\epsilon^\alpha}\frac{d}{ds}, \quad \frac{d^2}{dx^2} = \frac{1}{\epsilon^{2\alpha}}\frac{d^2}{ds^2}.$$

4. Substituting the derivatives in terms of s:

$$\epsilon \frac{1}{\epsilon^{2\alpha}}\frac{d^2 f_{in}}{ds^2} - \frac{1}{\epsilon^\alpha}\frac{df_{in}}{ds} + f_{in} = 0.$$

Simplify:

$$\epsilon^{1-2\alpha}\frac{d^2 f_{in}}{ds^2} - \epsilon^{-\alpha}\frac{df_{in}}{ds} + f_{in} = 0.$$

5. Determine Dominant Balance: To ensure all terms contribute to leading-order behavior, the powers of ϵ must balance. Compare the exponents: - The first term has $\epsilon^{1-2\alpha}$. - The second term has $\epsilon^{-\alpha}$. - The third term has ϵ^0.

If we require the first and second terms to balance, we get $1 - 2\alpha = -\alpha$, which gives:

$$\alpha = 1.$$

We can see that if $\alpha = 1$, then indeed the term in $+f_{in}$ is negligible compared with the other terms, so this solution is self-consistent.

6. Inner Equation: Substituting $\alpha = 1$, the inner equation becomes:

$$\frac{d^2 f_{in}}{ds^2} - \frac{df_{in}}{ds} = 0.$$

7. Solve the Inner Equation: Solve $\frac{d^2 f_{in}}{ds^2} - \frac{df_{in}}{ds} = 0$:

$$f_{in}(s) = C_2 + C_3 e^s.$$

8. Boundary Condition in the Boundary Layer: At $x = 1$ (or $s = 0$), $f(1) = 2$. Thus:

$$f_{in}(s = 0) = C_2 + C_3 = 2.$$

9. Prandtl's Matching Condition: Moving out of the boundary layer corresponds to $s \to -\infty$, so the matching condition is

$$\lim_{s\to-\infty} f_{in}(s) = \lim_{x\to 1} f_{out}(x) \to C_2 = e$$

Substituting into the boundary condition $C_2 + C_3 = 2$, we find:

$$C_3 = 2 - e.$$

10. Composite Expansion: The composite solution is:

$$f_{comp}(x) = f_{out}(x) + f_{in}(s) - (\text{common part}).$$

Substituting:

$$f_{comp}(x) = e^x + (e + (2-e)e^{\frac{x-1}{\epsilon}}) - e.$$

Simplify:

$$f_{comp}(x) = e^x + (2-e)e^{\frac{x-1}{\epsilon}}.$$

Example 3:

$$\epsilon \frac{d^2 f}{dx^2} + x\frac{df}{dx} + xf = 0 \tag{3.238}$$

with $f(-1) = e$, $f(1) = 2/e$. This time we had an internal boundary layer at $x = 0$, which means that we have 2 outer solutions on either side, and an inner solution in the vicinity of $x = 0$. The method of solution, however, is fairly similar.

- Find the solutions in each outer region (call them f_{out-} for the left-side ($x < 0$) and f_{out+} for the right-side ($x > 0$). Note that the integration constants can be different in each outer regions – find them by applying the relevant boundary condition in each case.
- Let $s = \frac{x}{\epsilon^\alpha}$ inside the boundary layer. Substitute in the governing equation. Identify the dominant balance, and deduce what α is.
- Show that the inner solution is of the form

$$f_{in}(s) = a \cdot \text{erf}\left(\frac{s}{\sqrt{2}}\right) + b \tag{3.239}$$

where erf denotes the error function, which is defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{3.240}$$

Find the values of the integration constants a and b by applying Prandtl's matching condition on either side.

- Try to create a composite expansion (you may need to define it differently for $x > 0$ and $x < 0$).
- Compare the composite expansion to the numerical solution for different values of ϵ

Solution: From the numerical solution we know that the function $\epsilon\frac{d^2f}{dx^2} + x\frac{df}{dx} + xf = 0$ has an internal boundary around $x = 0$, so the outer solution exists on both $x < 0$ and $x > 0$ sides.

The outer equation should be:

$$\frac{df_{out}}{dx} + f_{out} = 0$$

And the solution is

$$f_{outL}(x) = A_L e^{-x} \quad x < 0$$

or

$$f_{outR}(x) = A_R e^{-x} \quad x > 0$$

We can apply the two boundary conditions $f(-1) = e$ for A_L and $f(1) = 2/e$ for A_R, the solution can be easily found as: $A_L = 1$ and $A_R = 2$. Substituting it into our previous equation, we get the outer equation result:

$$f_{outL}(x) = e^{-x} \quad x > 0$$

$$f_{outR}(x) = 2e^{-x} \quad x > 0$$

Next, we can try to find the inner solution, ie, the solution when x is near 0. First we propose the rescaling $s = \frac{x}{\epsilon^\alpha}$, we have

$$x\frac{df}{dx} = \frac{\epsilon^\alpha s}{\epsilon^\alpha}\frac{df}{ds} = s\frac{df}{ds}$$

And substitute it into the original equation, which is:

$$\epsilon^{1-2\alpha}\frac{d^2f}{ds^2} + s\frac{df}{ds} + \epsilon^\alpha sf = 0$$

The term $\epsilon^\alpha sf$ in this equation is relatively smaller than others, we can just ignore this term, and the dominant balance becomes:

$$\epsilon^{1-2\alpha} = 1 \to \alpha = 1/2 \to s = \frac{x}{\sqrt{\epsilon}}$$

The boundary layer equation is then:

$$\frac{d^2f_{in}}{ds^2} + s\frac{df_{in}}{ds} = 0$$

To solve this one, let $\frac{dg}{ds} + sg = 0$, then have $dg/g = -sds$

$$\ln(g) = -\frac{s^2}{2} + Const$$

so

$$g(s) = Ke^{-s^2/2} = \frac{df}{ds}$$

We can integrate this to obtain

$$f_{in}(s) - f_{in}(0) = \int_0^s Ke^{-s'^2/2}ds' = \sqrt{\frac{\pi}{2}}K\mathrm{erf}(\frac{s}{\sqrt{2}})$$

In this equation, erf is the error function. And this function still left two unknown constants K and $f_{in}(0)$. To find their values, we need use the Prandtl's matching condition on *both* sides of the boundary layer. For the left side ($s \to -\infty$:

$$\lim_{x\to 0} A_L e^{-x} = \lim_{s\to -\infty} f_{in}(s)$$

Noting that $\lim_{x\to+\infty} \mathrm{erf}(x) = 1$, $\lim_{x\to-\infty} \mathrm{erf}(x) = -1$, we have

$$A_L = f_{in}(0) - K\sqrt{\frac{\pi}{2}} = 1$$

For the right side:

$$\lim_{x\to 0} A_R e^{-x} = \lim_{s\to +\infty} f_{in}(s)$$

$$A_R = f_{in}(0) + K\sqrt{\frac{\pi}{2}} = 2$$

Having the A_R and A_L, we can get the value of $f_{in}(0) = \frac{3}{2}$ and $K = \frac{1}{\sqrt{2\pi}}$, the inner equation becomes:

$$f_{in}(s) = \frac{3}{2} + \frac{1}{2}erf(\frac{s}{\sqrt{2}})$$

Finally, composite them, we have the separated solution.
For $x > 0$:

$$\begin{aligned} f_{comp} &= f_{out}(x) + f_{in}(s) - L \\ &= 2e^{-x} + \frac{3}{2} + \frac{1}{2}\mathrm{erf}(\frac{x}{\sqrt{2\epsilon}}) - 2 \\ &= 2e^{-x} - \frac{1}{2} + \frac{1}{2}\mathrm{erf}(\frac{x}{\sqrt{2\epsilon}}) \end{aligned} \tag{3.241}$$

And for $x < 0$:

$$f_{comp} = e^{-x} + \frac{1}{2} + \frac{1}{2}\mathrm{erf}(\frac{x}{\sqrt{2\epsilon}})$$

Note how in this case, the thickness of the boundary layer is $O(\epsilon^{1/2})$ instead of $O(\epsilon)$, as in the other two examples. This demonstrates that one must be careful, and always look for dominant balance to find the correct boundary layer properties.

Chapter 4

Calculus of Variations

Lecture edited by Yiqin, Jeremy and Janice

The final chapter of this course introduces an important technique in applied mathematics that is used in the field of optimization. In regular Calculus, we learned how to find the points at which a particular function of one or more variables achieves a minimum or a maximum. We do this by taking their derivatives and looking for stationary points. In this Chapter, we will generalize the idea to find the *functions* at which a certain *functional* achieves a minimum or a maximum. Let's first define what a functional is (in the context of Calculus of Variations), and see why it is related to optimization.

4.1 Functionals and their optimization

In the context of Calculus of Variations, we will define a **functional** as an operator that takes one or multiple functions (each of one or multiple variables), and returns a scalar number. Generally, these operators are integral operators over some domain.

Examples:

- The 'simplest' functional is the area under a curve over a domain $[a, b]$:

$$A[f(x)] = \int_a^b f(x)dx$$

  This functional takes a function f, and returns a scalar that is the signed area under the curve $y = f(x)$.

- We can also define a functional that is the length of the curve $y = f(x)$ over $[a, b]$:
  Solution:

$$L[f(x)] = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx$$

- Suppose we have a 2-dimensional flow field $\mathbf{u} = (u(x, y), v(x, y))$, the total kinetic energy of that flow field over some domain D is a functional of both u and v:

$$E[u(x, y), v(x, y)] = \int \int \frac{1}{2} \left(u^2(x, y) + v^2(x, y)\right) dxdy$$

- Suppose a car uses fuel at a rate that is proportional to the square of its velocity. Then if the car travels at velocity $v(t)$ between time $t = 0$ and time $t = T$, the amount of fuel consumed is:
  Solution:

$$F[v(t)] = C \int_0^T v(t)^2 dt$$

and so forth. We see that functionals that are integrals of one or more functions and their derivatives, and naturally come up all the time in applied mathematics.

Definition: If a functional is defined as an integral over some domain $[a, b]$ then the associated **Lagrangian** is its integrand. The Lagrangian is usually denoted as $\mathcal{L}$ (which stands for the L in Lagrangian, rather than implying that it is a linear operator), and is a function of the independent variables, of the functions, and of their derivatives.

Example:

- In the example of the length of a curve, the Lagrangian is:
  Solution:
$$\mathcal{L}(x, f, \frac{df}{dx}) = \sqrt{1 + (\frac{df}{dx})^2}$$

- In the example of flow kinetic energy the Lagrangian is: **Solution:**
$$\mathcal{L} = \frac{1}{2}\left(u^2(x, y) + v^2(x, y)\right)$$

Having defined functionals, we are often interested in *optimizing* (i.e. minimizing or maximizing) them. For example,

- if we fix $y(a)$ and $y(b)$, what is the shape of the curve $y = f(x)$ that takes the shortest path from the point $(a, y(a))$, to the point $(b, y(b))$?
- if we fix the duration of the car trip to be T, and assume that the car starts and stops with $v = 0$, what is the optimal way to drive it (i.e. to select $v(t)$) to minimize fuel consumption?

etc.

We see that optimizing functionals requires finding the *function* that minimizes or maximizes it. Solving this new type of optimization problem leads us to introduce *variational calculus* (also called Calculus of Variations).

4.2 Functional derivative of $H[f(x)]$

By analogy with regular Calculus, the first step to maximizing or minimizing functionals requires defining what their derivatives with respect to their input variable is. These are equivalently called *functional derivatives*, *variational derivatives*, or sometimes simply *variations*. As we are about to see, defining these derivatives is a little bit tricky.

Let us start, for simplicity, with functionals that only take a single input function f of a single independent variable x, namely $H[f(x)]$. Naively, and by analogy with regular Calculus, we would like say that the variational derivative of the functional $H[f(x)]$ with respect to f, evaluated at $f = f_0$ is defined as
$$\left.\frac{\delta H}{\delta f}\right|_{f=f_0} = \lim_{\epsilon \to 0} \frac{H[f_0 + \epsilon] - H[f_0]}{\epsilon} \tag{4.1}$$
The problem is that ϵ here really ought to be a function of x just like $f_0(x)$ is, so it is not completely clear what the limit would mean in this formula.

To make progress, let's note that if we had discretized the domain $[a, b]$ as
$$x_n = a + n\Delta x \text{ for } n = 0 \text{ to } n = N$$
with $\Delta x = (b - a)/N$ (so $x_0 = a$ and $x_N = b$), then $H[f(x)]$ could be viewed (approximately) as a multivariate function of $N + 1$ variables $\{f_n\}$ where $f_n = f(x_n)$.

Example: Consider the functional
$$H[f(x)] = \int_a^b x^2 f^2(x)dx = \int_a^b \mathcal{L}(x, f)dx \text{ with } \mathcal{L}(x, f) = x^2 f^2 \tag{4.2}$$

This can be discretized as a multivariate function using the trapezoidal rule:

$$H[f(x)] \simeq \Delta x \left(\frac{x_0^2}{2} f^2(x_0) + x_1^2 f^2(x_1) + ...x_{N-1}^2 f^2(x_{N-1}) + \frac{x_N^2}{2} f^2(x_N) \right)$$
$$\simeq \Delta x \left(\frac{x_0^2}{2} f_0^2 + x_1^2 f_1^2 + ...x_{N-1}^2 f_{N-1}^2 + \frac{x_N^2}{2} f_N^2 \right) \equiv h_{N+1}(f_0, f_1, ..., f_N) \tag{4.3}$$

where the $f_n = f(x_n)$ are the independent variables for h_{N+1} (and all the other terms are just 'coefficients' which depend on the discretization selected, and are constants). The subscript $N+1$ on the function h_{N+1} simply accounts for the fact it corresponds to the discretization with $N+1$ points, and is a function of $N+1$ variables. The approximation tends to the true functional in the limit of $N \to \infty$:

$$\lim_{N\to\infty} h_{N+1}(f_0, ..., f_N) = H[f(x)]$$

That being the case, we can use what we know of multivariate Calculus to create the 'derivative' of H as the limit of the 'derivative' of the multivariate function h_{N+1} as $N \to \infty$. To do this, first recall that for multivariate functions, we must be careful taking derivatives because they depend on the 'direction' taken. So instead of a derivative, the relevant quantity is the gradient of the function:

$$\nabla h_{N+1} = \left(\frac{\partial h_{N+1}}{\partial f_0}, ..., \frac{\partial h_{N+1}}{\partial f_N} \right).$$

Furthermore, we also recall that the infinitesimal change in h_{N+1} with respect to infinitesimal changes in the $N+1$ input variables $\{f_n\}$ (which can combined into the vector $\mathbf{f}$) is related to the gradient of h_{N+1} via

$$dh_{N+1} = h_{N+1}(\mathbf{f} + \mathbf{df}) - h_{N+1}(\mathbf{f}) \simeq \nabla h_{N+1} \cdot \mathbf{df} + h.o.t \simeq \sum_{n=0}^{N} \frac{\partial h_{N+1}}{\partial f_n} df_n + h.o.t \tag{4.4}$$

where $\mathbf{df}$ is an $N+1$ dimensional vector that records changes in each f_n, whose norm is assumed to be small.

If we take the limit of this expression as $N \to \infty$ we see that this would become

$$\lim_{N\to\infty} dh_{N+1} = H[f(x) + \epsilon(x)] - H[f(x)] = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{\partial h_{N+1}}{\partial f_n} \epsilon(x_n) + h.o.t$$

where we have constructed the function $\epsilon(x)$ to be such that $\epsilon(x_n) = df_n$. The infinite sum on the right-hand side tends to an integral as $N \to \infty$. To see this, let's look again at the same example: if h_{N+1} is defined by (4.3) then

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{\partial h_{N+1}}{\partial f_n} df_n = \lim_{N\to\infty} \Delta x \left(\frac{x_0^2}{2} 2 f_0 df_0 + x_1^2 2 f_1 df_1 + ... \right.$$
$$\left. + x_{N-1}^2 2 f_{N-1} df_{N-1} + \frac{x_N^2}{2} 2 f_N df_N \right) = \int_a^b 2x^2 f(x) \epsilon(x) dx \tag{4.5}$$

Furthermore, for this specific example we see that this can be rewritten as

$$H[f(x) + \epsilon(x)] - H[f(x)] = \int_a^b \frac{\partial \mathcal{L}}{\partial f} \epsilon(x) dx \tag{4.6}$$

since

$$\mathcal{L}(x, f) = x^2 f^2 \to \frac{\partial \mathcal{L}}{\partial f} = 2x^2 f$$

This example, while very simple, now allows us to see why the following definition of the functional derivative may make sense.

Definition: The functional derivative of $H[f(x)]$ with respect to the function f, at $f = f_0$, written as $\delta H/\delta f|_{f=f_0}$, satisfies

$$H[f_0(x) + \epsilon(x)] - H[f_0(x)] = \int_a^b \left.\frac{\delta H}{\delta f}\right|_{f=f_0} \epsilon(x)dx + h.o.t \tag{4.7}$$

for any smooth function $\epsilon(x)$ such that $|\epsilon(x)| \ll |f_0(x)|$.

We see, by comparing this expression with (4.4), that $\delta H/\delta f$ is simply the infinite-dimensional version of a gradient! Furthermore, this expression tells us *how* to find $\delta H/\delta f|_{f=f_0}$: first, compute $H[f_0(x) + \epsilon(x)] - H([f_0(x)]$, and rewrite this as an integral that involves $\epsilon(x)$. The *other* quantity in the integral is the variational derivative you are looking for.

Example 1: In the example above, we found that if $H[f(x)]$ is given by (4.2), then

$$H[f(x) + \epsilon(x)] - H[f(x)] = \int_a^b 2x^2 f(x)\epsilon(x)dx = \int_a^b \frac{\partial \mathcal{L}}{\partial f}\epsilon(x)dx \tag{4.8}$$

This shows that

$$\frac{\delta H}{\delta f} = 2x^2 f(x) = \frac{\partial \mathcal{L}}{\partial f} \tag{4.9}$$

While we obtained this result by discretizing the interval $[a, b]$, we can obtain it much more quickly using the definition above (now that we know why the definition makes sense). Indeed,

$$\begin{aligned} H[f(x) + \epsilon(x)] - H[f(x)] &= \int_a^b x^2(f(x) + \epsilon(x))^2 dx - \int_a^b x^2 f^2(x)dx \\ &= \int_a^b x^2(2f(x)\epsilon(x) + \epsilon^2(x))dx = \int_a^b 2x^2 f(x)\epsilon(x)dx + h.o.t \end{aligned} \tag{4.10}$$

which proves (4.9).

In fact, we can now prove a much more general result.

$$\text{If } H[f(x)] = \int_a^b \mathcal{L}(x, f)dx \text{ then } \frac{\delta H}{\delta f} = \frac{\partial \mathcal{L}}{\partial f} \tag{4.11}$$

Proof: Starting with our assumption, we follow the procedure of finding the functional derivative $\delta H/\delta f$:

- First compute $H[f_0(x) + \epsilon(x)] - H[f_0(x)]$.
- Second: rewrite this expression as an integral involving $\epsilon(x)$. and $\epsilon'(x)$
- Lastly use IBP to recast the result so the expression under the integral as it will be equal to $\int_a^b \left.\frac{\delta H}{\delta f}\right|_{f=f_0} \epsilon(x)dx$ which will then tell us what the variational derivative $\delta H/\delta f$ is.

We may rewrite the expression $H[f(x) + \epsilon(x)] - H[f(x)]$ as:

$$\begin{aligned} H[f(x) + \epsilon(x)] - H[f(x)] &= \int_a^b \mathcal{L}(x, f(x) + \epsilon(x))dx - \int_a^b \mathcal{L}(x, f(x))dx \\ &= \int_a^b \left[\mathcal{L}(x, f(x)) + \epsilon(x)\frac{\partial \mathcal{L}(x, f)}{\partial f} + \mathcal{O}(\epsilon(x)^2)\right] dx - \int_a^b \mathcal{L}(x, f(x))dx, \end{aligned}$$

where we substituted $\mathcal{L}(x, f(x) + \epsilon(x))$ with its Taylor expansion in $\epsilon(x)$. Note that the terms that are independent of $\epsilon(x)$ cancel, and so we're left with the following:

$$H[f(x) + \epsilon(x)] - H[f(x)] = \int_a^b \epsilon(x)\frac{\partial \mathcal{L}(x, f(x))}{\partial f}dx, \tag{4.12}$$

Comparing this with the definition of the variational derivative

$$\int_a^b \frac{\delta H}{\delta f}\epsilon(x)dx = \int_a^b \epsilon(x)\frac{\partial \mathcal{L}}{\partial f}dx.$$

shows that

$$\frac{\delta H}{\delta f} = \frac{\partial \mathcal{L}}{\partial f} \text{ when } \mathcal{L} = \mathcal{L}(x, f) \qquad \square \tag{4.13}$$

Lecture by Henry, Alyn and Alex

Example 2: Let's now compute the variational derivative of the functional

$$H[f(x)] = \int_a^b (f(x)^2 + f'(x)^2)dx \tag{4.14}$$

To do so, we compute

$$\begin{aligned} H[f(x)+\epsilon(x)] - H[f(x)] &= \int_a^b ((f+\epsilon)^2 + (f'+\epsilon')^2)dx - \int_a^b (f^2 + f'^2)dx \\ &= \int_a^b (2f\epsilon + \epsilon^2 + 2f'\epsilon' + \epsilon'^2)dx \\ &= 2\int_a^b (f(x)\epsilon(x) + f'(x)\epsilon'(x))dx + h.o.t \end{aligned}$$

This is not quite in the form we want: the $\epsilon'(x)$ term needs further work so the integral only contains $\epsilon(x)$. But this is exactly what integration by parts is for. We then get

$$H[f(x)+\epsilon(x)] - H[f(x)] = 2\int_a^b (f\epsilon - f''\epsilon)dx + 2[f'\epsilon]_a^b + h.o.t$$

Assuming that $f(a)$ and $f(b)$ are known at the end-points of the interval (which is usually going to be the case, see examples below), then these are *not* varied, and therefore $\epsilon(a) = \epsilon(b) = 0$. This concludes that

$$H[f(x)+\epsilon(x)] - H[f(x)] = \int_a^b 2(f-f'')\epsilon(x)dx + h.o.t \rightarrow \frac{\delta H}{\delta f} = 2(f - f'') \tag{4.15}$$

The same technique can be used to find the functional derivative of any general functional H of the form

$$H[f(x)] = \int_a^b \mathcal{L}(x, f, f')dx \tag{4.16}$$

Solution: We begin with:

$$H[f(x)+\epsilon(x)] - H[f(x)] = \int_a^b \mathcal{L}(x, f+\epsilon, \frac{df}{dx} + \frac{d\epsilon}{dx})dx - \int_a^b \mathcal{L}(x, f, \frac{df}{dx})dx \tag{4.17}$$

Using a Taylor series expansion, we can expand the RHS as:

$$\int_a^b \mathcal{L}(x,f,f')dx + \int_a^b \epsilon\frac{\partial\mathcal{L}}{\partial f}dx + \int_a^b \epsilon'\frac{\partial\mathcal{L}}{\partial f'}dx - \int_a^b \mathcal{L}(x,f,f')dx \tag{4.18}$$

$$= \int_a^b \epsilon\frac{\partial\mathcal{L}}{\partial f}dx + \int_a^b \epsilon'\frac{\partial\mathcal{L}}{\partial f'}dx \tag{4.19}$$

where we used the notation $\frac{df}{dx} = f'$ and $\frac{d\epsilon}{dx} = \epsilon'$ and ignored higher order terms in ϵ. In order for this expression to correspond with equation (4.7), we need it to be of the form:

$$\int_a^b \frac{\delta H}{\delta f}\epsilon(x)dx \tag{4.20}$$

The first term in our expression is okay, but the second term:

$$\int_a^b \epsilon'\frac{\partial\mathcal{L}}{\partial f'}dx \tag{4.21}$$

has an ϵ' that we want to turn in to an ϵ. The most obvious path forward is integration by parts with:

$$u = \frac{\partial\mathcal{L}}{\partial f'},\;\; du = \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial f'}\,dx \tag{4.22}$$

$$dv = \epsilon'\,dx,\;\; v = \epsilon \tag{4.23}$$

which allows us to write:

$$\int_a^b \epsilon' \frac{\partial \mathcal{L}}{\partial f'} dx = \left[\epsilon \frac{\partial \mathcal{L}}{\partial f'}\right]_a^b - \int_a^b \epsilon \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f'} dx \tag{4.24}$$

If the function is 'pinned' and $x = a$ and $x = b$ (so $f(a)$ and $f(b)$ are given and known), then we conclude $\epsilon(a) = \epsilon(b) = 0$ since the function is not varying at either endpoint. Taking this to be the case, we have:

$$\int_a^b \epsilon' \frac{\partial \mathcal{L}}{\partial f'} dx = \cancelto{0}{\left[\epsilon \frac{\partial \mathcal{L}}{\partial f'}\right]_a^b} - \int_a^b \epsilon \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f'} dx = -\int_a^b \epsilon \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f'} dx \tag{4.25}$$

Putting all of this together:

$$H[f(x)+\epsilon(x)] - H[f(x)] = \int_a^b \epsilon \frac{\partial \mathcal{L}}{\partial f} dx - \int_a^b \epsilon \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f'} dx \tag{4.26}$$

$$= \int_a^b \epsilon \left[\frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f'}\right] dx \tag{4.27}$$

Which gives us:

$$\frac{\delta H}{\delta f} = \frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f'} \tag{4.28}$$

Similarly let compute the variational derivative of a general functional H given by

$$H[f(x)] = \int_a^b \mathcal{L}(x, f, f', f'')dx \tag{4.29}$$

where $f''(x) = d^2 f/dx^2$.

Solution:

Similarly to the last example, we begin with:

$$H[f(x)+\epsilon(x)] - H[f(x)] = \int_a^b \mathcal{L}(x, f+\epsilon, \frac{df}{dx} + \frac{d\epsilon}{dx}, \frac{d^2f}{dx^2} + \frac{d^2\epsilon}{dx^2})dx - \int_a^b \mathcal{L}(x, f, \frac{df}{dx}, \frac{d^2f}{dx^2})dx \tag{4.30}$$

Again using a Taylor Expansion the RHS can be reformulated as

$$\int_a^b \mathcal{L}\, dx + \int_a^b \epsilon \frac{\partial \mathcal{L}}{\partial f} dx + \int_a^b \epsilon' \frac{\partial \mathcal{L}}{\partial f'} dx + \int_a^b \epsilon'' \frac{\partial \mathcal{L}}{\partial f''} dx - \int_a^b \mathcal{L}\, dx \tag{4.31}$$

$$= \int_a^b \epsilon \frac{\partial \mathcal{L}}{\partial f} dx + \int_a^b \epsilon' \frac{\partial \mathcal{L}}{\partial f'} dx + \int_a^b \epsilon'' \frac{\partial \mathcal{L}}{\partial f''} dx \tag{4.32}$$

We can use integration by parts in the ϵ' term just as we did in the previous example to get

$$\int_a^b \epsilon' \frac{\partial \mathcal{L}}{\partial f'} dx = -\int_a^b \epsilon \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f'} dx$$

But for the ϵ'' term we must apply the method twice:

$$\int_a^b \epsilon'' \frac{\partial \mathcal{L}}{\partial f''}\, dx : \tag{4.33}$$

$$u = \frac{\partial \mathcal{L}}{\partial f''},\ du = \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f''}\, dx \tag{4.34}$$

$$dv = \epsilon''\, dx,\ v = \epsilon' \tag{4.35}$$

$$\Rightarrow \left[\epsilon' \frac{\partial \mathcal{L}}{\partial f''}\right]_a^b \to 0 - \int_a^b \epsilon' \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f''}\, dx \tag{4.36}$$

$$u = \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f''},\ du = \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial f''}\, dx \tag{4.37}$$

$$dv = \epsilon'\, dx,\ v = \epsilon \tag{4.38}$$

$$\Rightarrow -\left(\left[\epsilon \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f''}\right]_a^b \to 0 - \int_a^b \epsilon \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial f''}\, dx\right) \tag{4.39}$$

$$= \int_a^b \epsilon \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial f''}\, dx \tag{4.40}$$

So equation (4.34) simplifies to:

$$\int_a^b \epsilon \frac{\partial \mathcal{L}}{\partial f}\, dx + \int_a^b \epsilon' \frac{\partial \mathcal{L}}{\partial f'}\, dx + \int_a^b \epsilon'' \frac{\partial \mathcal{L}}{\partial f''}\, dx = \int_a^b \epsilon \left[\frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial f''}\right)\right] dx \tag{4.41}$$

And thus:

$$H[f(x) + \epsilon(x)] - H[f(x)] = \int_a^b \epsilon \left[\frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial f''}\right)\right] dx \tag{4.42}$$

Which yields the final result:

$$\frac{\delta H}{\delta f} = \frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial f''}\right) \tag{4.43}$$

We therefore see that computing functional derivatives of functionals of a single function of a single variable is as easy as integration by parts, and can easily be generalized to include even higher-order terms!

Later in this series of lectures we will extend these definitions to functionals that take multiple functions of multiple variables.

4.3 Optimizing functionals of the form $H[f(x)]$

Let's now learn to use these functional derivatives to optimize simple functionals of the form $H[f(x)]$. The interpretation of a functional derivative as the 'continuum' version of a gradient suggests that we can easily find the function that optimizes a functional $H[f(x)]$ by setting

$$\frac{\delta H}{\delta f} = 0, \tag{4.44}$$

which will yield an ODE (see below) and solving for f. This is the equivalent of optimizing a multivariate function h simply by setting $\nabla h = 0$ and finding the points at which this happens.

In the case where $H[f(x)] = \int_a^b \mathcal{L}(x, f, f')dx$, for instance,

$$\frac{\delta H}{\delta f} = 0 \to \frac{\partial \mathcal{L}}{\partial f} = \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f'}$$

This equation is called the **Euler-Lagrange equation** associated with the optimization of H. Note that this equation is an ODE whose solution is the 'optimal' function f.

Let's see how this works out through various examples.

Example 1: Find the curve $y = f(x)$ with the shortest path from (a, y_a) to (b, y_b), assuming a, b, y_a and y_b are given.

To solve this problem, we first construct the functional that is the length of the path:

$$L[f(x)] = \int_a^b \sqrt{1 + f'^2(x)}dx \tag{4.45}$$

To minimize it, we first calculate the functional derivative:

$$\frac{\delta H}{\delta f} = \frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial f'} = 0 - \frac{d}{dx}\left[\frac{f'}{\sqrt{1+f'^2}}\right] = 0 \tag{4.46}$$

This equation simply says that

$$\frac{f'}{\sqrt{1+f'^2}} = C \tag{4.47}$$

where C is some constant, and a simple solution to that equation is $f'(x) = K$ (some other constant), so

$$f(x) = Kx + D \tag{4.48}$$

which is clearly the equation of a line. Finally, we apply boundary conditions: the curve goes from (a, y_a) to (b, y_b) , so we have $f(a) = y_a$ and $f(b) = y_b$. The line that satisfies this is

$$y = f(x) = \frac{y_b - y_a}{b - a}(x - a) + y_a \tag{4.49}$$

Example 2: Find the curve $y = f(x)$ from (a, y_a) to (b, y_b) assuming a, b, y_a and y_b are given with $b > a$ and $y_b < y_a$, along which an object would slide under gravity (but without friction), starting from rest, in the least possible time. This is called **the Brachistochrone problem**. To do that

- Explain why the time it takes is given by

  $$T[f(x)] = \int_a^b \frac{dl}{v}$$

  where $dl = \sqrt{1 + f'^2}dx$ is the arc length, and $v(t)$ satisfies

  $$\frac{1}{2}mv^2 + mgy = C \tag{4.51}$$

  where C is a constant to be determined.

- Derive and solve (analytically or numerically) the corresponding Euler-Lagrange equation when $a = 0$, $b = 1$, $y_a = 1$ and $y_b = 0$.

Solution: First we consider the functional which we are optimizing, here we wish we minimize the time it takes to go from point a to point b:

$$\begin{aligned} T[f(x)] &= \int_a^b dt \text{ where } dt \text{ is the time it takes to travel length } dl \\ &= \int_a^b \frac{dl}{v} \text{ by definition} \end{aligned}$$

Now we wish to find v in relation to $f(x)$. To find v, we use the principle of conservation of energy:

$$\begin{aligned}
\text{kinetic energy} + \text{potential energy} &= \text{constant} \\
\frac{1}{2}mv^2 + mgf(x) &= C \\
&= mgy_a \\
\frac{1}{2}mv^2 + mgf(x) &= mgy_a \\
\frac{1}{2}v^2 + gf(x) &= gy_a \\
v^2 &= 2g(y_a - f(x)) \\
v &= \sqrt{2g(y_a - f(x))}
\end{aligned}$$

Now, by definition of arclength, we have $dl = \sqrt{1 - f'(x)^2}dx$, and we construct T:

$$T[f(x)] = \int_a^b \frac{\sqrt{1 + f'(x)^2}}{\sqrt{2g(y_a - f(x))}}dx$$

To minimize T, we consider its Lagrangian denoted $\mathcal{L}_T$ and compute the variational derivative $\frac{\delta T}{\delta f}$:

$$\begin{aligned}
\mathcal{L}_T &= \frac{\sqrt{1 + f'(x)^2}}{\sqrt{2g(y_a - f(x))}} \\
\frac{\delta T}{\delta f} &= \frac{\partial \mathcal{L}_T}{\partial f} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}_T}{\partial f'}\right) \\
&= \frac{\sqrt{1 + f'^2}}{2}\frac{1}{(y_a - f)^{3/2}} - \frac{f'^2}{2(y_a - f)^{3/2}}\frac{1}{(1 + f'^2)^{1/2}} - \frac{1}{(y_a - f)^{1/2}}\frac{f''}{(1 + f'^2)^{3/2}} \\
&= \frac{\sqrt{1 + f'^2}}{2(y_a - f)^{3/2}} - \frac{f'^2}{2(y_a - f)^{3/2}(1 + f'^2)^{1/2}} - \frac{f''}{(1 + f'^2)^{3/2}(y_a - f)^{1/2}} \\
&= \frac{1 + f'^2}{2(y_a - f)^{3/2}} - \frac{f'^2}{2(y_a - f)^{3/2}} - \frac{f''}{(1 + f'^2)(y_a - f)^{1/2}} \\
&= \frac{1 + f'^2}{2} - \frac{f'^2}{2} - \frac{f''(y_a - f)}{(1 + f'^2)} \\
&= 1 - 2\frac{f''(y_a - f)}{(1 + f'^2)}
\end{aligned}$$

To minimize T, we set $\frac{\delta T}{\delta f} = 0$, and find $1 + f'^2 = 2f''(y_a - f)$. Now we numerically solve the ODE with boundary conditions $y_a = f(a)$ and $y_b = f(b)$, shown in 4.1.
To find the solution, we consider $a = 0, b = 1, y_a = 1$ and $y_b = 0$

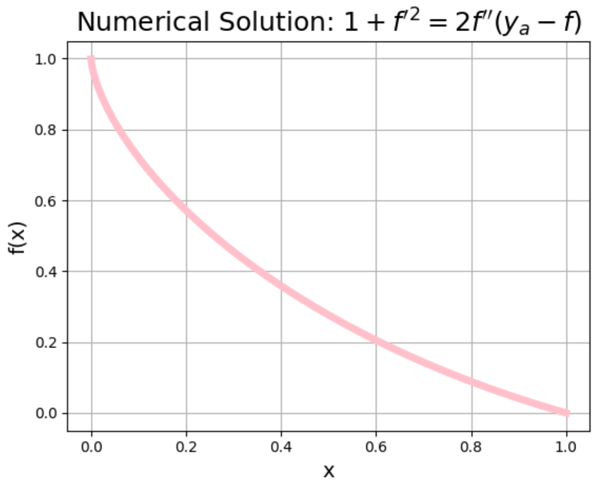

Figure 4.1: Numerical solution of the Euler-Lagrange equation resulting from Example 2, where a = 0, b = 1, $y_a = 1$ and $y_b = 0$

4.4 Constrained optimization of functionals of the form $H[f(x)]$

Lecture edited by Sean and Arthur

In this lecture, we extend the concepts learned in the previous lecture to look at *constrained* optimization. As we shall see, we can once again easily extend tools of multivariate Calculus to do so. First of all, let's review how to solve constrained optimization problems for multivariate functions.

4.4.1 The method of Lagrange multipliers in multivariate Calculus

Suppose we wish to optimize (minimize or maximize) a multivariate function $f(\mathbf{x})$ where $\mathbf{x}$ is an N-dimensional vector, *subject to the constraint* that $c(\mathbf{x}) = 0$. Here, the functions $f(\mathbf{x})$ and $c(\mathbf{x})$ are known, and we are looking for the optimal point $\mathbf{x} = (x_1, x_2, ..., x_N)$, noting that there may be several solutions.

The constraint $c(\mathbf{x}) = 0$ generally forms an $N-1$ dimensional subspace (sometimes called manifold) in the N-dimensional space. For instance:

- The constraint $x = 1$, which we can write as $c(x) = x - 1 = 0$ forms a line on the 2D Cartesian space
- The constraint $r = 1$, which we can write $c(x, y, z) = \sqrt{x^2 + y^2 + z^2} - 1 = 0$ forms the surface of a sphere in 3D space.

It is easy to convince oneself with a simple figure that maximizing or minimizing f subject to the constraint $c(\mathbf{x}) = 0$ boils down to finding the points where the isocontours of f are tangent to the curve/surface/etc. described by the constraint.

Isocontours of f being tangent to the constraint curve (which is the 0 isocontour of c), simply implies that ∇f is parallel to ∇c (because ∇f and ∇c are perpendicular to the isocontours of f and c, respectively).

To find the points where this happens, we therefore find all of the solutions of the system of equations formed by

$$c(\mathbf{x}) = 0 \tag{4.52}$$
$$\nabla f = \mu \nabla c \tag{4.53}$$

for some μ (that is also to be found as part of the solution). The scalar μ is called the *Lagrange Multiplier*. Let's look at a few examples.

Example 1: What is the shortest distance between the origin and the plane $z = 1 - x - y$?

Solution: The quantity we wish to minimize is

$$f(x, y, z) = x^2 + y^2 + z^2$$

The constraint is given by the function

$$c(x, y, z) = 1 - x - y - z = 0$$

The minimum is achieved when $\nabla f = \mu \nabla c$ which implies

$$\begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \mu \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \tag{4.54}$$
$$x = -\frac{\mu}{2} \tag{4.55}$$
$$y = -\frac{\mu}{2} \tag{4.56}$$
$$z = -\frac{\mu}{2} \tag{4.57}$$

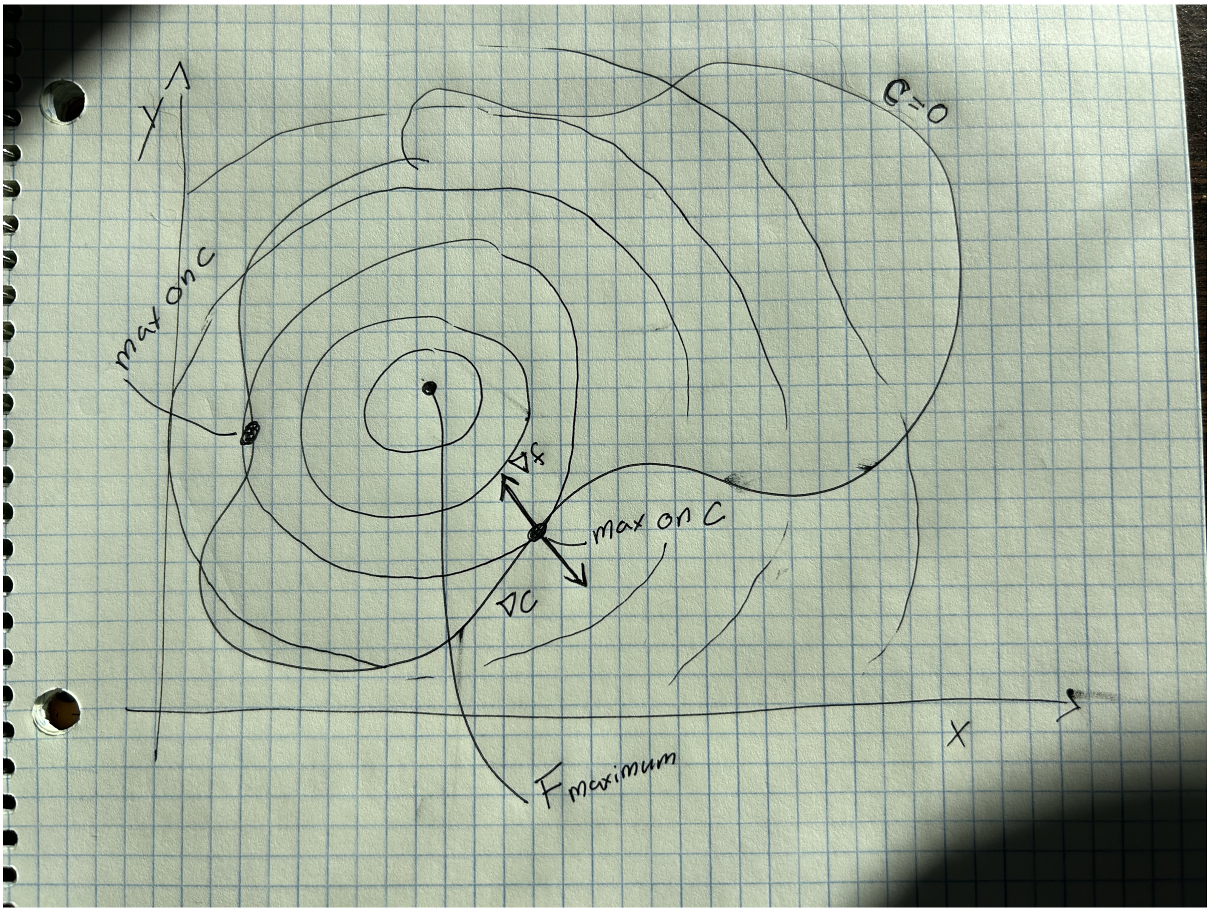

Figure 4.2: In this examples, the contours of f are centered on its assumed maximum. The gradient of f is perpendicular to the contours. The constraint is the contour $C = 0$, and the ∇C is perpendicular to this contour. f achieves maxima on C at points where the two gradients are parallel.

We need to ensure it fits the constraint

$$1 + \frac{\mu}{2} + \frac{\mu}{2} + \frac{\mu}{2} = 0 \tag{4.58}$$

$$\mu = -\frac{2}{3} \tag{4.59}$$

$$x = \frac{1}{3} \tag{4.60}$$

$$y = \frac{1}{3} \tag{4.61}$$

$$z = \frac{1}{3} \tag{4.62}$$

Note that we can also try to solve problems with multiple constraints – we simply add each of them with its own Lagrange multiplier. However, in this case it's important to check that the intersection of the constraints exists before diving into the mathematical problem – otherwise we can waste a lot of time.

Example 2: Find the stationary points of $f(x, y, z) = x^3 + y^3 + z^3$ subject to the two constraints $x^2 + y^2 + z^2 = 1$ and $x + y + z = 0$. Here, we note that the plane $x + y + z = 0$ does intersect with the sphere $x^2 + y^2 + z^2 = 1$ on a circle in 3D space centered at the origin, so the question has a solution.

Solution: Here we want to find the stationary points of $f(x, y, z) = x^3 + y^3 + z^3$, subject to constraints $c_1(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$ and $c_2(x,y,z) = x + y + z = 0$. We solve simultaneously

$$\nabla f = \lambda_1 \nabla c_1 + \lambda_2 \nabla c_2 \tag{4.63}$$
$$c_1(x,y,z) = 0 \tag{4.64}$$
$$c_2(x,y,z) = 0 \tag{4.65}$$

This is equivalent to

$$3x^2 = 2\lambda_1 x + \lambda_2 \tag{4.66}$$
$$3y^2 = 2\lambda_1 y + \lambda_2 \tag{4.67}$$
$$3z^2 = 2\lambda_1 z + \lambda_2 \tag{4.68}$$

and the two constraints. Adding these three equations together, and using the constraints, we immediately find that $3 = 3\lambda_2$, so

$$\lambda_2 = 1 \tag{4.69}$$

Then, x, y and z all satisfy the same equation, namely

$$3x^2 - 2\lambda_1 x - 1 = 0$$

which has solutions

$$x = \frac{\lambda_1 \pm \sqrt{3 + \lambda_1^2}}{3} \tag{4.70}$$

These two solutions have different signs (if one is positive, the other is negative). We also know that y and z can take the same 2 values only. So we either have

$$x = y \to z = -2x = -2y \tag{4.71}$$
$$x = z \to y = -2x = -2z \tag{4.72}$$
$$y = z \to x = -2y = -2z \tag{4.73}$$

Let's consider the first scenario. Suppose

$$z = \frac{\lambda_1 + \sqrt{3 + \lambda_1^2}}{3}, \quad x = y = \frac{\lambda_1 - \sqrt{3 + \lambda_1^2}}{3} \tag{4.74}$$

Then, $x + y + z = 0$ implies

$$\frac{\lambda_1 + \sqrt{3 + \lambda_1^2}}{3} + 2\frac{\lambda_1 - \sqrt{3 + \lambda_1^2}}{3} = 0 \to \lambda_1 = \frac{\sqrt{3 + \lambda_1^2}}{3} \tag{4.75}$$

which implies $8\lambda_1^2 = 3$ and has the solution

$$\lambda_1 = \sqrt{\frac{3}{8}}$$

from which we can reconstruct (x,y,z):

$$x = y = -\frac{2}{3}\sqrt{3/8}, z = \frac{4}{3}\sqrt{3/8} \tag{4.76}$$

We can also instead assume

$$x = y = \frac{\lambda_1 + \sqrt{3 + \lambda_1^2}}{3}, \quad z = \frac{\lambda_1 - \sqrt{3 + \lambda_1^2}}{3} \tag{4.77}$$

Then, $x + y + z = 0$ implies

$$2\frac{\lambda_1 + \sqrt{3 + \lambda_1^2}}{3} + \frac{\lambda_1 - \sqrt{3 + \lambda_1^2}}{3} = 0 \to \lambda_1 = -\frac{\sqrt{3 + \lambda_1^2}}{3} \tag{4.78}$$

from which we deduce that

$$\lambda_1 = -\sqrt{\frac{3}{8}} \tag{4.79}$$

and we can again reconstruct (x,y,z) from there:

$$x = y = \frac{2}{3}\sqrt{3/8}, z = -\frac{4}{3}\sqrt{3/8} \tag{4.80}$$

The other solutions are found by simple permutations of x, y and z.

4.4.2 Constrained optimization of functionals with an integral constraint

Drawing again on the analogy that views functional derivatives as a continuum version of the gradient, we can see that in order to optimize a functional $H[f(x)]$, subject to the constraint that some other functional $C[f(x)] = 0$, can be done by requiring at the same time that

$$C[f(x)] = 0 \tag{4.81}$$
$$\frac{\delta H}{\delta f} = \mu \frac{\delta C}{\delta f} \tag{4.82}$$

Let's see a few examples.

Example 1: Dido's problem Show that the only possible (smooth) closed, simply connected curve of given perimeter P that maximizes a given area A is a circle.

- Consider a point O somewhere inside the curve, and define the equation of the curve in polar coordinates as $r = f(\theta)$ where $\theta \in [0, 2\pi]$, and r is the distance from the point O to the curve.
- Write down the area and the perimeter of the curve as functionals of $r(\theta)$.
- Construct the optimization problem, and solve it.

Solution:
First, note that our shape must be convex. If it were concave, we could simply cut across the convex part, which decrease the perimeter. We define the functional H which represents the area to be maximized as

$$H[f] = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{f^2(\theta)}{2} d\theta$$

The constraint that needs to be satisfied is:

$$C[f] = \int_0^{2\pi} f(\theta) d\theta - P = 0$$

The optimal shape is obtained when

$$\frac{\delta H}{\delta f} = \mu \frac{\delta C}{\delta f}$$

which becomes

$$f = \mu \times 1$$

To find μ we use the constraint $C[f] = 0$

$$\int_0^{2\pi} f(\theta) d\theta = P \tag{4.83}$$
$$\int_0^{2\pi} \mu d\theta = 2\pi\mu \tag{4.84}$$
$$\mu = \frac{P}{2\pi} \tag{4.85}$$

So, the optimal shape is circle of radius $P/2\pi$.

Lecture edited by Jeremy, Janice and Moein

Example 2 (harder): The catenary curve. A catenary curve is the shape that a hanging chain or a hanging necklace of a given length takes when held at two points at the same height. Physically speaking, that is the shape adopted by the chain as it minimizes potential energy. To find it requires solving a constrained optimization problem involving functionals. To see this, suppose that the shape of the chain is given by $y = f(x)$. We expect that shape to be symmetric, and so assume that it is held at $y = 0$ at both $x = \pm a$. With this, the boundary conditions are $f(a) = 0$ and $f'(0) = 0$, and we only need to find the solution on the interval $[0, a]$.

Then we know that

- The length of the chain is fixed and equal to L_C, therefore

$$L_C = \int dl = 2\int_0^a \sqrt{1+f'^2}dx$$

- The potential energy of the chain is equal to

$$P[f(x)] = \int \rho g f dl = 2\int_0^a \rho g f\sqrt{1+f'^2}dx$$

  where we will assume here that ρ is a constant linear mass density.

We therefore want to minimize $P[f(x)]$ subject to the constraint

$$C[f(x)] = 2\int_0^a \sqrt{1+f'^2}dx - L_C = \int_0^a \left[2\sqrt{1+f'^2} - \frac{L_C}{a}\right] dx = 0 \tag{4.86}$$

To do so is conceptually similar to Dido's problem, but the mathematics are significantly harder.

- Show that the optimization problem yields the ODE

$$\frac{f''}{1+f'^2} = \frac{1}{f-\mu}$$

  where μ is the Lagrange multiplier for the problem.

- By multiplying both sides by f', show that this can be integrated once to yield $f'(x) = \sqrt{C(f-\mu)^2 - 1}$ where C is an integration constant.

- Use a change of variable, and Wolfram Alpha (or tables of integrals) to show that $f(x) = \mu + C^{-1/2}\cosh(K + C^{1/2}x)$

- Apply the boundary conditions, and the integral constraint, to find C, K and μ. Note that the actual values of these constants can only be obtained by solving a transcendental equation numerically (but the relationship between the constants can be written analytically)

Solution :

We note that this is a constrained optimization problem; i.e. we are not minimizing over all possible functions, which would lead us to solve for:

$$\frac{\delta P}{\delta f} = 0.$$

Instead, we are looking for solutions of the constrained problem, which satisfies:

$$\frac{\delta P}{\delta f} = \mu\frac{\delta C}{\delta f} \tag{4.87}$$

subject to the constraint: $C[f] = 0$.

Letting $\mathcal{L}_P = 2\rho g f\left[1+f'^2\right]^{1/2}$ and $\mathcal{L}_C = 2\left[1+f'^2\right]^{1/2} - \frac{L_C}{a}$, we have:

$$\frac{\delta P}{\delta f} = \mu \frac{\delta C}{\delta f} = \frac{\partial \mathcal{L}_p}{\partial f} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}_P}{\partial f'}\right) = \mu \left[\frac{\partial \mathcal{L}_C}{\partial f} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}_C}{\partial f'}\right)\right]$$
$$\implies \rho g \left[1+f'^2\right]^{1/2} - \frac{d}{dx}\left(\rho g f f' \left[1+f'^2\right]^{-1/2}\right) = \mu \frac{d}{dx}\left[1+f'^2\right]^{-1/2} f'$$
$$\implies \left[1+f'^2\right]^{1/2} - f'^2\left[1+f'^2\right]^{-1/2} - \underbrace{f\frac{d}{dx}\left(f'\left[1+f'^2\right]^{-1/2}\right)}_{\left(f''+f'^2f''[1-f'^2]^{-1}\right)\cdot[1+f'^2]^{-1/2}} = -\frac{\mu}{\rho g}\frac{d}{dx}\left(f'\left[1+f'^2\right]^{-1/2}\right)$$
$$\implies 1 - \frac{f'^2}{[1+f'^2]} - \frac{f\left(f'' - f'^2 f''\left[1+f'^2\right]^{-1}\right)}{[1+f'^2]} = -\frac{\mu}{\rho g}\frac{f'' - f'^2 f''\left[1+f'^2\right]^{-1}}{[1+f'^2]}$$
$$\implies \frac{1}{1+f'^2} = \left(f - \frac{\mu}{\rho g}\right)\left[\frac{1 - f'^2\left[1+f'^2\right]^{-1}}{1+f'^2}\right] f''$$
$$\implies 1 = \left(f - \frac{\mu}{\rho g}\right)\left[\frac{1+f'^2-f'^2}{1+f'^2}\right] f'' = \left(f - \frac{\mu}{\rho g}\right)\left[\frac{1}{1+f'^2}\right] f''$$

which finally yields the ODE we are looking for:

$$\frac{f''}{1+f'^2} = \frac{1}{f - \frac{\mu}{\rho g}}. \tag{4.88}$$

Multiply both sides by $2 \times f'(x)$:

$$\frac{2f''f'}{1+f'^2} = \frac{2f'}{f - \frac{\mu}{\rho g}}. \tag{4.89}$$

Notice that both sides can be rewritten as a derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(1+f'^2)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(2\ln\left|f - \frac{\mu}{\rho g}\right|\right). \tag{4.90}$$

Integrating both sides with respect to x, we get:

$$\ln(1+f'^2) = 2\ln\left|f - \frac{\mu}{\rho g}\right| + C_1, \tag{4.91}$$

where C_1 is a constant of integration. Exponentiating both sides:

$$1 + f'^2 = C\left(f - \frac{\mu}{\rho g}\right)^2, \tag{4.92}$$

where $C = e^{C_1}$ is a new constant. Solving for f'^2:

$$f'^2 = C\left(f - \frac{\mu}{\rho g}\right)^2 - 1. \tag{4.93}$$

Taking the square root on both sides:

$$f'(x) = \pm\sqrt{C\left(f - \frac{\mu}{\rho g}\right)^2 - 1}. \tag{4.94}$$

We choose the positive sign since we are integrating from 0 to a, and f' is positive in this region.

Let $v = \sqrt{C}\left(f - \frac{\mu}{\rho g}\right)$, so $v'(x) = \sqrt{C} f'(x)$ and $f'(x) = \sqrt{v^2 - 1}$. Separating variables:

$$\frac{\mathrm{d}v}{\sqrt{v^2-1}} = \sqrt{C}\,\mathrm{d}x. \tag{4.95}$$

The integral on the left is standard and evaluates as:

$$\int \frac{\mathrm{d}v}{\sqrt{v^2-1}} = \cosh^{-1}(v). \tag{4.96}$$

Thus:

$$\cosh^{-1}(v) = \sqrt{C}x + K, \tag{4.97}$$

where K is a constant of integration. Solving for v:

$$v(x) = \cosh\left(K + \sqrt{C}x\right). \tag{4.98}$$

Substituting back $v = \sqrt{C}\left(f - \frac{\mu}{\rho g}\right)$, we have:

$$\sqrt{C}\left(f - \frac{\mu}{\rho g}\right) = \cosh\left(K + \sqrt{C}x\right). \tag{4.99}$$

Finally, solving for $f(x)$:

$$f(x) = \frac{1}{\sqrt{C}}\cosh\left(K + \sqrt{C}x\right) + \frac{\mu}{\rho g}. \tag{4.100}$$

This is the desired solution.

We now apply the boundary conditions, and the integral constraint, to find C, K and μ.

First we apply $f'(0) = 0$ (the symmetry condition):

$$f'(x) = \sinh(\sqrt{C}x + k) \tag{4.101}$$

$$f'(0) = \sinh(k) = 0 \tag{4.102}$$

$$\rightarrow k = 0 \tag{4.103}$$

Next we apply $f(a) = 0$:

$$\frac{1}{\sqrt{C}}\cosh(\sqrt{C}a) + \frac{\mu}{\rho g} = 0 \tag{4.104}$$

Finally we apply the integral constraint:

$$L_C = 2\int_0^a \sqrt{1 + f'^2}dx$$

$$L_C = 2\int_0^a \sqrt{1 + \sinh(\sqrt{C}x)^2}dx \tag{4.105}$$

Making the substitution $1 + \sinh^2(\sqrt{C}x) = \cosh^2(\sqrt{C}x)$:

$$L_C = 2\int_0^a \sqrt{\cosh(\sqrt{C}x)^2}dx = 2\int_0^a \cosh(\sqrt{C}x)dx = \left[\frac{2}{\sqrt{C}}\sinh(\sqrt{C}x)\right]_a^0 = \frac{2}{\sqrt{C}}\sinh(\sqrt{C}a) \tag{4.106}$$

Note that L_c and a are given by the problem, so we can solve (4.106) using, say, a Newton Solve algorithm (see Matlab routine) to obtain C, then solve (4.104) to obtain μ and complete the problem.

Finally, note that this is not the only way of enforcing constraints. Later in this Chapter we will also learn how to impose *pointwise* constraints.

4.5 More functions and more variables

Lecture edited by Sean, Arthur and Yiqin

In this section, we will learn how to optimize functionals that involve more derivatives, and/or more functions. Let's begin by computing more functional derivatives.

4.5.1 Functional derivatives for $H[f(\mathbf{x})]$, $H[\mathbf{f}(x)]$, and $H[\mathbf{f}(\mathbf{x})]$

Single function of multiple variables: Let's start with functionals of functions of multiple variables, namely $H[f(\mathbf{x})]$, where $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. For simplicity, let's assume that the Lagrangian depends *at most* on the partial derivatives of f with respect to the x_i, but not on second derivatives (it is easy to extend the method to that case, however), so

$$H[f(\mathbf{x})] = \int_D \mathcal{L}\left(\mathbf{x}, f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}\right) dx_1 \ldots dx_N \tag{4.107}$$

Let's compute the functional derivative. As before, we form dH and Taylor expand the result in the limit of small $|\epsilon|$

$$\begin{aligned} H[f(\mathbf{x}) + \epsilon(\mathbf{x})] - H[f(\mathbf{x})] &= \int_D \mathcal{L}\left(\mathbf{x}, f+\epsilon, \frac{\partial (f+\epsilon)}{\partial x_1}, \ldots, \frac{\partial (f+\epsilon)}{\partial x_N}\right) dx_1 \ldots dx_N \\ &\quad - \int_D \mathcal{L}\left(\mathbf{x}, f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}\right) dx_1 \ldots dx_N \\ &= \int_D \left[\epsilon \frac{\partial \mathcal{L}}{\partial f} + \frac{\partial \epsilon}{\partial x_1} \frac{\partial \mathcal{L}}{\partial(\partial f/\partial x_1)} + \ldots + \frac{\partial \epsilon}{\partial x_N} \frac{\partial \mathcal{L}}{\partial(\partial f/\partial x_N)}\right] dx_1 \ldots dx_N + h.o.t \end{aligned} \tag{4.108}$$

The first term is already in the right form, and for the other terms, we integrate them by parts in the variables x_i, one by one, so

$$\begin{aligned} &H[f(\mathbf{x}) + \epsilon(\mathbf{x})] - H[f(\mathbf{x})] \\ &= \int_D \left[\epsilon \frac{\partial \mathcal{L}}{\partial f} - \epsilon \frac{\partial}{\partial x_1} \frac{\partial \mathcal{L}}{\partial(\partial f/\partial x_1)} - \ldots - \epsilon \frac{\partial}{\partial x_N} \frac{\partial \mathcal{L}}{\partial(\partial f/\partial x_N)}\right] dx_1 \ldots dx_N + h.o.t \\ &= \int_D \epsilon \frac{\delta H}{\delta f} dx_1 \ldots dx_N + h.o.t \end{aligned} \tag{4.109}$$

provided

$$\frac{\delta H}{\delta f} = \frac{\partial \mathcal{L}}{\partial f} - \frac{\partial}{\partial x_1} \frac{\partial \mathcal{L}}{\partial(\partial f/\partial x_1)} - \ldots - \frac{\partial}{\partial x_N} \frac{\partial \mathcal{L}}{\partial(\partial f/\partial x_N)} \tag{4.110}$$

Multiple functions of a single variable: Let's now consider the functional

$$H[\mathbf{f}(x)] = H[f_1(x), \ldots, f_N(x)] = \int_a^b \mathcal{L}(x, f_1, \ldots, f_N, f_1', \ldots, f_N') dx \tag{4.111}$$

where again for simplicity we assume that $\mathcal{L}$ does not depend on any higher order derivatives (but we can easily extend the result to that case if necessary).

This time, because H is a functional of multiple functions f_i, we can take the variation of H with respect to each f_i separately: there will be N such variations,

$$\frac{\delta H}{\delta f_1}, \ldots \frac{\delta H}{\delta f_N} \tag{4.112}$$

Let's compute $\delta H/\delta f_i$, which assumes only f_i varies, but all the other functions are fixed. It is defined such that

$$dH = H[f_1, \ldots, f_i + \epsilon, \ldots, f_N] - H[\mathbf{f}] = \int_a^b \epsilon(x) \frac{\delta H}{\delta f_i} dx \tag{4.113}$$

It is easy to show that

$$\frac{\delta H}{\delta f_i} = \frac{\partial \mathcal{L}}{\partial f_i} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial f_i'} \tag{4.114}$$

(just as it would be if H were a function of f_i alone).

Multiple functions of multiple variables: Finally, if we have multiple functions of multiple variables, we simply combine these two results. For the sake of simplicity, let's just assume that we have two functions f_1 and f_2 of two variables x and y, and that

$$H[f_1, f_2] = \int_D \mathcal{L}\left(x, y, f_1, f_2, \frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}\right) dxdy \tag{4.115}$$

We then find that

$$\frac{\delta H}{\delta f_1} = \frac{\partial \mathcal{L}}{\partial f_1} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial f_1/\partial x)}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial(\partial f_1/\partial y)}\right) \tag{4.116}$$

$$\frac{\delta H}{\delta f_2} = \frac{\partial \mathcal{L}}{\partial f_2} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial f_2/\partial x)}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial(\partial f_2/\partial y)}\right) \tag{4.117}$$

4.5.2 Multidimensional optimization problems

Let's now use what we have learned so far to solve a few interesting problems.

Example 1: Consider a particle in 2D space, with coordinates $x(t)$ and $y(t)$, evolving in a potential field $\Phi(x, y)$. We define the *action* $\mathcal{S}$ (it is traditionally denoted with that symbol for some reason) as the integrated difference between the particle kinetic and potential energy over time.

$$\mathcal{S}[x(t), y(t)] = \int_0^T \left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\left(\frac{dy}{dt}\right)^2 - \Phi(x, y)\right] dt \tag{4.118}$$

What equations are obtained my minimizing the action?

Solution: Let $\mathcal{L} = \frac{1}{2}(x'^2 + y'^2) - \Phi(x, y)$ where primes denote derivatives with respect to time.

$$\frac{\delta S}{\delta x} = 0 = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x'} = -\frac{\partial \Phi}{\partial x} - x'' \tag{4.119}$$

$$\frac{\delta S}{\delta y} = 0 = \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial y'} = -\frac{\partial \Phi}{\partial y} - y'' \tag{4.120}$$

This then recovers Newton's law:

$$(x, y)'' = -\nabla\Phi \tag{4.121}$$

Example 2: Consider the set of all possible incompressible two-dimensional flow field $[u(x, y), v(x, y)]$ on the unit square, satisfying $u = 0$ and $\partial v/\partial x = 0$ on the vertical $x = 0$ and $x = 1$ boundaries, and $v = 0$ and $\partial u/\partial y = 0$ on the horizontal $y = 0$ and $y = 1$ boundaries. Among all of these flow fields we would like to find the one that has the smallest possible total dissipation

$$D = \nu \int_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] dxdy \tag{4.122}$$

(where ν is a constant called viscosity) for a fixed total kinetic energy

$$E = \frac{1}{2}\int_D [u^2 + v^2] dxdy \tag{4.123}$$

To solve this problem, we note that there are two constraints: the fact that the kinetic energy is known and the fact that everywhere in D we want to impose incompressibility

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.124}$$

The first constraint is a regular integral constraint, but the second is a *pointwise* constraint that we want to impose at every (x, y). To do so we simply have a Lagrange Multiplier that is a function of (x, y) rather than a constant. The functional we want to optimize is therefore

$$H[u,v] = D[u,v] - \mu \int_D \left(\frac{u^2+v^2}{2} - E\right) dxdy - \int_D \lambda(x,y)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy \tag{4.125}$$

We can now find and solve the system of *PDEs* that yields the solution with least dissipation.

Solution: Let $\partial u/\partial x = u_x$, $\partial u/\partial y = u_y$, etc. Then

$$\frac{\delta D}{\delta u} = 0 - \frac{\partial}{\partial x}(2\nu u_x) - \frac{\partial}{\partial y}(2\nu u_y) = -2\nu u_{xx} - 2\nu u_{yy} = -2\nu\nabla^2 u$$

and similarly

$$\frac{\delta D}{\delta v} = -2\nu\nabla^2 v$$

Then if we define:

$$C_1 = \mu \int_D \left(\frac{u^2+v^2}{2} - E\right) dxdy \tag{4.126}$$

$$C_2 = \int_D \lambda(x,y)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy \tag{4.127}$$

we have

$$\frac{\delta C_1}{\delta u} = \mu u - 0 = \mu u, \quad \frac{\delta C_1}{\delta v} = \mu v \tag{4.128}$$

$$\frac{\delta C_2}{\delta u} = 0 - \frac{\partial \lambda}{\partial x} - 0 = -\lambda_x, \quad \frac{\delta C_2}{\delta v} = -\lambda_y \tag{4.129}$$

In summary, the EL equations are:

$$-\mu u = -\lambda_x + 2\nu\nabla^2 u \tag{4.130}$$

$$-\mu v = -\lambda_y + 2\nu\nabla^2 v \tag{4.131}$$

which looks a lot like linearized Navier-Stokes equations, with the Lagrange multiplier function $\lambda(x, y)$ playing the role of pressure! [PG: Please add the derivation of the PDE, and I will add its solution]

4.6 Formula Sheet

4.6.1 PDEs and Green's Functions

Sturm-Liouville problems revolve around:

$$\mathcal{L}(u) = \frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = -\lambda w(x)u \tag{4.132}$$

With u, v solutions to (4.132):

$$\langle u, v\rangle = \int_{x_a}^{x_b} u(x)v(x)w(x)dx \tag{4.133}$$

To get in to SL form:

$$r(x) = \frac{p(x)}{a(x)}, p(x) = \exp\left(\int \frac{b(x)}{a(x)}dx\right) \tag{4.134}$$

Wave Equation:

$$f_{tt} = c^2\nabla^2 f$$

Wave gen: [PG: I do not agree with this solution, it is not 'general']

$$f(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\right]\sin\left(\frac{n\pi x}{L}\right) \tag{4.135}$$

Heat Equation:

$$f_t = k\nabla^2 f$$

Heat gen: [PG: I do not agree with this solution, it is not 'general']

$$f(x,t) = c_0 + \sum_{n=1}^{\infty} c_n\cos\left(\frac{n\pi x}{L}\right)e^{-t/\tau_n} \tag{4.136}$$

$$\tau_n = \frac{L^2}{Dn^2\pi^2} \tag{4.137}$$

Schrodinger Equation:

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t)$$

4.6.2 Asymptotics

Linsted-Poincaré is used when nonlinearity causes change in period of oscillator:

$$f(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + \cdots \tag{4.138}$$

$$\tau = t(1 + a_1\epsilon + \cdots),\ \ \frac{d}{dt} = (1 + \epsilon a_0 + \cdots)\frac{\partial}{\partial\tau} \tag{4.139}$$

$$\frac{d^2}{dt^2} = (1 + \epsilon a_0 + \cdots)^2\frac{\partial^2}{\partial\tau^2} \tag{4.140}$$

When amplitude changes, use **multiple scales**:

$$f(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + \cdots$$

$$t_s = \epsilon t,\ t_f = t,\ \frac{d}{dt} = \epsilon\frac{\partial}{\partial t_s} + \frac{\partial}{\partial t_f}$$

$$\frac{d^2}{dt^2} = \epsilon^2\frac{\partial^2}{\partial t_s^2} + \frac{\partial^2}{\partial t_f^2} + 2\epsilon\frac{\partial^2}{\partial t_s\partial t_f}$$

$$\frac{\partial|A|e^{i\theta}}{\partial t_s} = i|A|e^{i\theta}\frac{\partial\theta}{\partial t_s} + \frac{\partial|A|}{\partial t_s}e^{i\theta}$$

When period and amplitude change, then we can use **WKB**:

$$f(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + \cdots \tag{4.141}$$

$$t_s = t,\ t_f = \frac{g(t)}{\epsilon},\ \ \frac{d}{dt} = \frac{\partial}{\partial t_s} + \frac{g'(t)}{\epsilon}\frac{\partial}{\partial t_f} \tag{4.142}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t_s^2} + \frac{g''}{\epsilon}\frac{\partial}{\partial t_f} + 2\frac{g'}{\epsilon}\frac{\partial^2}{\partial t_s t_f} + \frac{g'^2}{\epsilon^2}\frac{\partial^2}{\partial t_f^2} \tag{4.143}$$

(Note: You could use different t_f/t_s depending on the problem).

Boundary Layer Problems

[PG: I do not agree with this description]

Consider a differential equation of the form:

$$\epsilon f'' + f' + f = 0$$

for $x \in [a,b]$ with a boundary layer at x^*.

- Remind yourself where the boundary layer actually is.
- Find the outer solution via $\epsilon = 0$. Identify the integration constant by applying the appropriate boundary condition.
- Define the rescaled variable in the boundary layer as $s = (1-x)/\epsilon^\alpha$. (Note that you could also define $s = (x-1)/\epsilon^\alpha$ but then s would be negative everywhere).
- Write the ODE in s, and use arguments of dominant balance to explain why the inner balance is

$$\epsilon\frac{d^2 f_{in}}{ds^2} = \frac{df_{in}}{ds} \tag{4.144}$$

- Solve the inner equation and apply the relevant boundary condition in the boundary layer to eliminate one of the integration constants.
- Find the remaining integration constant by applying Prandtl's matching condition:

$$\lim_{s\to\pm\infty} f_{in}(s) = d_{in} = \lim_{x\to x^*} f_{out}(x) = c_{out} = \text{Limit}$$

- Construct the composite expansion:

$$f(t) = f_{\text{inner}} + f_{\text{outer}} - \text{Limit}$$

4.6.3 Calculus of Variations:

Define the functional:

$$H[f(x)] = \int_a^b \mathcal{L}(x, f, f', f'')dx \tag{4.145}$$

To find (min/max)ima:

$$\frac{\delta H}{\delta f} = 0 \implies \frac{\partial\mathcal{L}}{\partial f} = \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial f'} - \frac{d^2}{dx^2}\frac{\partial\mathcal{L}}{\partial f''} \tag{4.146}$$

If we have a constraint of the form:

$$C[f(x)] = 0 \tag{4.147}$$

Then we can optimize subject to the constraint with:

$$\frac{\delta H}{\delta f} = \mu\frac{\delta C}{\delta f} \tag{4.148}$$

Formula for n independent variables, 1 dependent variable, and single derivatives:

$$\frac{\partial\mathcal{L}}{\partial f} - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\frac{\partial\mathcal{L}}{\partial f_i}\right] \tag{4.149}$$

where $f_i = \frac{\partial f}{\partial x_i}$.

4.6.4 Some Trig Identities

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

$$\cos(a)\sin(b) = \frac{1}{2}\left[\sin(a+b) - \sin(a-b)\right]$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$

$$2\sin(x) = e^{ix} - e^{-ix},\ 2\cos(x) = e^{ix} + e^{-ix}$$

4.6.5 Laplace Transforms

$$\mathscr{L}[\delta(t-a)](s) = H(a)e^{-sa}, \qquad \mathscr{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathscr{L}[f(ct)] = \frac{1}{c}F\left(\frac{s}{c}\right), \qquad \mathscr{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$$

$$\mathscr{L}[f'(x)] = sF(s) - f(0), \qquad \mathscr{L}[f''(x)] = s^2F(s) - sf(0) - f'(0)$$

Convolution Theorem, $h(t) = \mathscr{L}^{-1}\left(\mathscr{L}[f]\mathscr{L}[g]\right)$:

$$h(t) = \int_0^t f(u)g(t-u)du = \int_0^t f(t-u)g(u)du$$

4.6.6 ODEs

Given the ODE of the form

$$\frac{dy}{dt} + p(t)y = g(t)$$

Use the integrating factor $\mu = \exp(\int p(t)dt)$.

Given an ODE of the form:

$$ay'' + by' + cy = g(t)$$

Where a, b, c are constant, solve the homogeneous case and then use the table of undetermined coefficients:

$g(t)$	guess for y_p
$ae^{\beta t}$	$Ae^{\beta t}$ unless $e^{\beta t}$ is sol. of homogen. eq.
$a\cos(\beta t)$ or $b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$ unless $e^{i\beta t}$ is sol. of homogen. eq.
nth poly	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$

If you reach a problem that cannot be solved analytically, search:

`https://www.wolframalpha.com/input?i= EQUATION`

where EQUATION is the ODE you need to solve.

4.6.7 Miscellaneous

Ernest Rutherford was the son of James Rutherford, a farmer, and his wife Martha Thompson, originally from Essex, England. Ernest was born near Nelson, New Zealand. Rutherford's mother was a schoolteacher. His first name was mistakenly spelt 'Earnest' when his birth was registered.

John Dalton was born in England in 1766, ten years before the U.S. Declaration of Independence was signed. His family lived in a small thatched cottage. As a small child, John worked in the fields with his older brother, and helped his father in the shop where they wove cloth. Although they had enough to eat, they were poor. Most poor boys at that time received no education, but John was lucky to attend a nearby school. In 1766, only about one out of every 200 people could read.

If people do not realize that mathematics is simple, it is only because they do not realize how complicated life is.

On 4 July 1934, at the Sancellemoz Sanatorium in Passy, France at the age of 66, Marie Curie died. The cause of her death was given as aplastic pernicious anaemia, a condition she developed after years of exposure to radiation through her work. She left two daughters, Irene (born 1898) and Eve (born 1904).