Homework 5

AM213A

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Part 1: Coding Problems

For the following coding problems please navigate to the `code` directory and run the following:

```
cd code
make clean
make
```

This will compile the c program and build the `main` executable.

Householder reduction to Hessenburg Form (Problem 1)

The module `hessenburg.c` contains the function `hessenburg()`, which implements Algorithm 26.1 from the book by Trefethen and Bau. This function takes a square symmetric matrix $\mathbf{A}$ as input and converts it into Hessenberg form, resulting in a tridiagonal matrix $\mathbf{T}$.

The file `a_one.dat` contains the data for the following matrix:

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{bmatrix} \tag{1}$$

We use the following command:

```
./main hessenberg a_one.dat
```

This produces the following tridiagonal matrix, printed to the console:

$$\mathbf{T} = \begin{bmatrix} 5.000000 & -4.242641 & 0.000000 & -0.000000 \\ -4.242641 & 6.000000 & 1.414214 & -0.000000 \\ -0.000000 & 1.414214 & 5.000000 & -0.000000 \\ -0.000000 & -0.000000 & -0.000000 & 2.000000 \end{bmatrix} \tag{2}$$

QR algorithm (Problem 2)

Perform a QR algorithm on the following Matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{3}$$

Without Shift

In the module `qr.c`, the function `QRnoshift()` implements Algorithm 28.1, the "Pure" QR Algorithm, from page 211 of the book by Trefethen and Bau. This function first converts the input matrix **A** into tridiagonal Hessenberg form. It then iteratively computes the QR factorization of this tridiagonal matrix using Householder factorization, as implemented in Homework Assignment 4.

The function then computes the matrix multiplication $\mathbf{R} \times \mathbf{Q}$ and extracts the subdiagonal elements into a column vector $\mathbf{x}$ of size $(m-1) \times 1$. The 2-norm of this column vector is computed and compared against a threshold value close to machine precision, 1×10^{-15}.

We use the following command:

```
./main qrnoshift a_two.dat
```

Which prints the following diagonal matrix to the console:

$$\mathbf{D} = \begin{bmatrix} 3.732051 & -0.000000 & 0.000000 \\ -0.000000 & 2.000000 & -0.000000 \\ 0.000000 & -0.000000 & 0.267949 \end{bmatrix} \tag{4}$$

Thus the eigenvalues of the matrix **A** are approximately $3.73, 2.0, 0.27$.

With Shift

The `QRshift` function takes in a matrix **A** assumed to be square and symmetric and converts the matrix in a diagonal matrix **D** containing the eigenvalues of **A** along the diagonal elements by using the QR Algorithm with shift and deflation as described in the class notes, page 112.

The function performs the following actions on the matrix **A**.

1. Converts **A** into tridiagonal Hessenberg Form.
2. Defines a square matrix **B** initially the same size as **A** where the size decrements by 1 until the final 2×2 matrix.
3. Set a value μ equal to the last diagonal entry of **B**.
4. Perform the following actions on B,
5. $$\mathbf{QR} = \mathbf{B} - \mu\mathbf{I} \tag{5}$$
6. $$\mathbf{B} = \mathbf{RQ} + \mu\mathbf{I} \tag{6}$$
7. Compute the minimum between the error $|\lambda_{n-1} - \lambda_n|$ and $|\lambda_{n-1} + \lambda_n|$ and check if it is above 1×10^{-15}. If it is, repeat steps 3 through 7, if it is below then go back to step 2 and decrement the size of **B** by one and perform steps 3-7 on this new matrix **B**.

We use the following command,

```
./main qrshift a_two.dat
```

Which prints the following matrix to the console:

$$\mathbf{D} = \begin{bmatrix} 3.732051 & -0.000000 & -0.000000 \\ 0.000000 & 2.000000 & -0.000000 \\ 0.000000 & 0.000000 & 0.267949 \end{bmatrix} \tag{7}$$

Comparison between shift and noshift

Interestingly we can compare the number of allocations it takes to converge to the diagonal matrix output by both algorithms by running the program with valgrind. By using the following command line call

```
valgrind ./main qrnoshift a_two.dat
```

shows that the QR algorithm with no shift used $1,762$ allocations and

```
valgrind ./main qrshift a_two.dat
```

the QR algorithm with shift and deflation used 252 allocations a $7\times$ decrease in the number of iterations it took to converge to the eigenvalues down to the same level of machine accuracy.

Inverse Iteration Algorithm (Problem 3)

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & -3 & 1 & 5 \\ 3 & 1 & 6 & -2 \\ 4 & 5 & -2 & -1 \end{bmatrix} \tag{8}$$

the data file associated with this matrix I have defined here to be `a_three.dat`.

Write a program of the inverse iteration to calculate the corresponding eigenvectors.

Inverse Iteration Algorithm

In the module `eigen.c` we needed a function that computes the inverse of a matrix. So I wrote a function called `invIP()` whose input is a square $\mathbf{A}$ symmetric matrix and transforms this matrix into its inverse by the following steps:

1. Computes the QR decomposition of the matrix using the householder reflector factorization method.

$$\mathbf{QR} = \mathbf{A} \tag{9}$$

2. Computes the inverse of the upper triangular matrix $\mathbf{R}$ by iterating backwards along the columns of the matrix and setting the diagonal elements $r_{i,i}^{-1} = \frac{1}{r_{i,i}}$ and the off diagonal elements $r_{j,i}^{-1} = -\frac{1}{r_{j,j}\sum_{k=j+1}^{i}} r_{j,k} * r_{k,i}^{-1}$for $j = i-1, \cdots, 0$
3. Computes $\mathbf{A}^{-1} = \mathbf{R}^{-1}\mathbf{Q}^T$

Finally, the `inverseIteration()` algorithm takes in a matrix $\mathbf{A}$ and an initial guess for an eigenvalue μ and follows the Inverse Iteration algorithm in the class notes on page 98.

The algorithm

1. Computes $\mathbf{B} = (\mathbf{A} - \mu\mathbf{I})^{-1}$.
2. Initializes the error $\|r\| = \infty$ and the vector $\mathbf{x}$ of random values between $[-1, 1]$ and then normalizes the vector $\mathbf{x}$ such that $\|\mathbf{x}\| = 1$.
3. do while $\|\mathbf{r}\| > \epsilon$ where $\epsilon = 1 \times 10^{-15}$
    a. Compute

$$\mathbf{y} = \frac{\mathbf{Bx}}{\|\mathbf{Bx}\|} \tag{10}$$

b. Set $\mathbf{r} = \mathbf{y} - \mathbf{x}$

c. Set $\mathbf{x} = \mathbf{y}$

d. Set $error = \|\mathbf{r}\|$

4. Print eigenvalue, eigenvector pair

We make the following command line call to the compiled function,

```
./main inverseiter a_three.dat
```

Which produces the following output to console:

For the matrix $\mathbf{A}$ we have the following eigenvalue, eigenvector pairs.

$$\lambda_1 = -8.028600 \tag{11}$$

$$\mathbf{v}_1 = \begin{bmatrix} -0.263462 \\ -0.659041 \\ 0.199634 \\ 0.675573 \end{bmatrix} \tag{12}$$

$$\lambda_2 = 7.932900 \tag{13}$$

$$\mathbf{v}_2 = \begin{bmatrix} 0.560145 \\ 0.211633 \\ 0.776708 \\ 0.195382 \end{bmatrix} \tag{14}$$

$$\lambda_3 = 5.668900 \tag{15}$$

$$\mathbf{v}_3 = \begin{bmatrix} 0.378703 \\ 0.362419 \\ -0.537935 \\ 0.660199 \end{bmatrix} \tag{16}$$

$$\lambda_4 = -1.573200 \tag{17}$$

$$\mathbf{v}_4 = \begin{bmatrix} -0.688048 \\ 0.624123 \\ 0.259801 \\ 0.263750 \end{bmatrix} \tag{18}$$

Part 2: Theory Problems

Problem 1

Consider the Householder matrix defined by

$$\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} \tag{19}$$

Part a

For any nonzero vector $\mathbf{v}$, the matrix is orthogonal and symmetric.

Proof. We need to show that the matrix $\mathbf{H}$ is orthogonal,

$$\mathbf{H}\mathbf{H}^T = \mathbf{H}^T\mathbf{H} = \mathbf{I} \tag{20}$$

and symmetric,

$$\mathbf{H}^T = \mathbf{H} \tag{21}$$

given that the vector $\mathbf{v}$ is nonzero.

Let us show that $\mathbf{H}^T = \mathbf{H}$,

$$\begin{aligned}
\mathbf{H}^T &= \left(\mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)^T && (22)\\
&= \mathbf{I} - 2\frac{\left(\mathbf{v}\mathbf{v}^T\right)^T}{\left(\mathbf{v}^T\mathbf{v}\right)^T} && (23)\\
&= \mathbf{I} - 2\frac{\left(\mathbf{v}^T\right)^T\mathbf{v}^T}{\mathbf{v}^T\left(\mathbf{v}^T\right)^T} && (24)\\
&= \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} && (25)\\
&= \mathbf{H} && (26)
\end{aligned}$$

Therefore $\mathbf{H} = \mathbf{H}^T$, the householder matrix is guaranteed to be symmetric.

Let us start with the (19) to show that $\mathbf{H}\mathbf{H}^T = \mathbf{H}^T\mathbf{H} = \mathbf{I}$. Since we just proved that the householder matrix is symmetric, then it must also be true that $\mathbf{H}^T\mathbf{H} = \mathbf{H}^2$.

$$\begin{aligned}
\mathbf{H} &= \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} && (27)\\
\mathbf{H}^T\mathbf{H} = \mathbf{H}^2 &= \left(\mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)^2 && (28)\\
&= \mathbf{I}^2 - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} + 4\left(\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)^2 && (29)\\
&= \mathbf{I} - 4\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} + 4\left(\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)^2 && (30)
\end{aligned}$$

If we can show that the matrix $\mathbf{P} = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$ is idempotent, or that $\mathbf{P} = \mathbf{P}^2$ then we know that the householder matrix $\mathbf{H}$ is an orthogonal matrix.

Let us compute $\mathbf{P}^2$

$$\left(\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)^2 = \frac{\mathbf{v}\left(\mathbf{v}^T\mathbf{v}\right)\mathbf{v}^T}{\left(\mathbf{v}^T\mathbf{v}\right)^2} \tag{31}$$

$$= \left(\mathbf{v}^T\mathbf{v}\right)\frac{\mathbf{v}\mathbf{v}^T}{\left(\mathbf{v}^T\mathbf{v}\right)^2} \tag{32}$$

$$= \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} \tag{33}$$

Therefore,

$$\mathbf{H}^T\mathbf{H} = \mathbf{H}^2 = \mathbf{I} \tag{34}$$

□

Part b

Let $\mathbf{a}$ be any non-zero vector and let $\mathbf{v} = \mathbf{a} + \alpha\mathbf{e}_1$, where $\alpha = \text{sign}(a_{11})\|\mathbf{a}\|_2$. Show that $\mathbf{H}\mathbf{a} = -\alpha\mathbf{e}_1$ by direct calculation.

Solution

Starting from the definition $\mathbf{H}$

$$\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} \tag{35}$$

$$\mathbf{H}\mathbf{a} = \left(\mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)\mathbf{a} \tag{36}$$

$$= \mathbf{a} - 2\frac{\mathbf{v}\mathbf{v}^T\mathbf{a}}{\mathbf{v}^T\mathbf{v}} \tag{37}$$

Let us plug in the definition for $\mathbf{v}$ into $\mathbf{v}\mathbf{v}^T\mathbf{a}$.

$$\mathbf{v}\mathbf{v}^T\mathbf{a} = \left(\mathbf{a} + \alpha\mathbf{e}_1\right)\left(\mathbf{a}^T + \alpha\mathbf{e}_1^T\right)\mathbf{a} \tag{38}$$

$$= \mathbf{a}\mathbf{a}^T\mathbf{a} + \alpha\mathbf{a}\mathbf{e}_1^T\mathbf{a} + \alpha\mathbf{e}_1\mathbf{a}^T\mathbf{a} + \alpha^2\mathbf{e}_1\mathbf{e}_1^T\mathbf{a} \tag{39}$$

To proceed let us acknowlege some facts,

$$\mathbf{a}^T\mathbf{a} = \|\mathbf{a}\|_2^2 \tag{40}$$

$$\mathbf{e}_1^T\mathbf{a} = \mathbf{a}^T\mathbf{e}_1 = a_1 \tag{41}$$

$$\alpha^2 = \left(\text{sign}(a_1)\|\mathbf{a}\|_2\right)^2 = \|\mathbf{a}\|_2^2 \tag{42}$$

$$\mathbf{e}_1^T\mathbf{e}_1 = 1 \tag{43}$$

We are left with

$$\mathbf{v}\mathbf{v}^T\mathbf{a} = \mathbf{a}\|\mathbf{a}\|_2^2 + \alpha\mathbf{a}a_1 + \alpha\mathbf{e}_1\|\mathbf{a}\|_2^2 + \alpha^2\mathbf{e}_1a_1 \tag{44}$$

$$= \mathbf{a}\left(\|\mathbf{a}\|_2^2 + \alpha a_1\right) + \alpha\mathbf{e}_1\left(\|\mathbf{a}\|_2^2 + \alpha a_1\right) \tag{45}$$

Let us compute $\mathbf{v}^T\mathbf{v}$

$$
\begin{aligned}
\mathbf{v}^T\mathbf{v} &= \left(\mathbf{a}^T + \alpha\mathbf{e}_1^T\right)\left(\mathbf{a} + \alpha\mathbf{e}_1\right) && (46)\\
&= \mathbf{a}^T\mathbf{a} + \alpha\mathbf{a}^T\mathbf{e}_1 + \alpha\mathbf{e}_1^T\mathbf{a} + \alpha^2\mathbf{e}_1^T\mathbf{e}_1 && (47)\\
&= \|\mathbf{a}\|_2^2 + \alpha a_1 + \alpha a_1 + \|\mathbf{a}\|_2^2 && (48)\\
&= 2\left(\|\mathbf{a}\|_2^2 + \alpha a_1\right) && (49)
\end{aligned}
$$

continuing from (37)

$$
\begin{aligned}
\mathbf{H}\mathbf{a} &= \mathbf{a} - 2\frac{\mathbf{a}\left(\|\mathbf{a}\|_2^2 + \alpha a_1\right) + \alpha\mathbf{e}_1\left(\|\mathbf{a}\|_2^2 + \alpha a_1\right)}{2\left(\|\mathbf{a}\|_2^2 + \alpha a_1\right)} && (50)\\
&= \mathbf{a} - \mathbf{a} - \alpha\mathbf{e}_1 && (51)\\
&= -\alpha\mathbf{e}_1 && (52)
\end{aligned}
$$

Part c

Determine $\mathbf{v}$ and α that transforms

$$
\mathbf{H}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}\alpha\\0\\0\\0\end{bmatrix} \qquad (53)
$$

Solution

Let $\mathbf{a} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

We want to construct a vector $\mathbf{v} = \mathbf{a} + \alpha\mathbf{e}_1$ such that $\mathbf{H}\mathbf{a} = -\alpha\mathbf{e}_1$.

this means we need to choose an α that satisfies these conditions. Notice that matrix multiplication by a vector containing all ones means that the vector produced is simply the sum of all the elements along the rows of the matrix.

Let us compute alpha by the formula $\alpha = \text{sign}(a_1)\|\mathbf{a}\|_2$

$$
\begin{aligned}
\alpha &= \text{sign}(a_1)\|\mathbf{a}\|_2 && (54)\\
&= \sqrt{4} = 2 && (55)
\end{aligned}
$$

Let us choose $\alpha = -2$

Let

$$
\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{56}
$$

$$
= \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{57}
$$

Let us compute $\mathbf{v}\mathbf{v}^T$

$$
\mathbf{v}\mathbf{v}^T = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 1 \end{bmatrix} \tag{58}
$$

$$
= \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \tag{59}
$$

Let us compute $\mathbf{v}^T\mathbf{v}$

$$
\mathbf{v}^T\mathbf{v} = \begin{bmatrix} -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{60}
$$

$$
= 4 \tag{61}
$$

Let us compute $\mathbf{H}$

$$
\mathbf{H} = I - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} \tag{62}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \tag{63}
$$

$$
= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{64}
$$

Finally let us compute $\mathbf{Ha}$

$$\mathbf{Ha} = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \tag{65}$$

$$= \frac{1}{2}\begin{bmatrix}4\\0\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\0\\0\end{bmatrix} \tag{66}$$

Thus we have shown that with an $\alpha = -2$ and $\mathbf{v} = \mathbf{a} + \alpha\mathbf{e}_1$ where $\mathbf{a} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

That $\mathbf{Ha} = -\alpha\mathbf{e}_1$

Part d

Given the vector $\mathbf{a} = \begin{bmatrix}2\\3\\4\end{bmatrix}$, specify a Householder transformation that annihilates the third component of $\mathbf{a}$.

Solution

We would like to find a householder matrix $\mathbf{H}$ such that

$$\mathbf{H}\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}\star\\\star\\0\end{bmatrix} \tag{67}$$

To do this we will construct a column vector $\mathbf{v}$ and a corresponding householder matrix $\mathbf{H}$.

Let

$$\mathbf{a} = \begin{bmatrix}0\\3\\4\end{bmatrix} \tag{68}$$

$$\alpha = \text{sign}(a_2)\|\mathbf{a}\|_2 \tag{69}$$

$$= \text{sign}(3)\sqrt{3^2 + 4^2} \tag{70}$$

$$= 5 \tag{71}$$

Let

$$\begin{aligned}
\mathbf{v} &= \mathbf{a} + \alpha \mathbf{e}_2 && (72)\\
&= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} && (73)\\
&= \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} && (74)
\end{aligned}$$

Let us compute $\mathbf{v}\mathbf{v}^T$

$$\begin{aligned}
\mathbf{v}\mathbf{v}^T &= \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & 8 & 4 \end{bmatrix} && (75)\\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 64 & 32 \\ 0 & 32 & 16 \end{bmatrix} && (76)
\end{aligned}$$

Let us compute $\mathbf{v}^T\mathbf{v}$

$$\begin{aligned}
\mathbf{v}^T\mathbf{v} &= \begin{bmatrix} 0 & 8 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} && (77)\\
&= 80 && (78)
\end{aligned}$$

Let us compute $\mathbf{H}$

$$\begin{aligned}
\mathbf{H} &= I - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} && (79)\\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{40} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 64 & 32 \\ 0 & 32 & 16 \end{bmatrix} && (80)\\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.6 & -0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix} && (81)
\end{aligned}$$

Finally let us compute (67) and show that the third component has been annihilated.

$$\begin{aligned}
\mathbf{H} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.6 & -0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} && (82)\\
&= \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix} && (83)
\end{aligned}$$

Part e

What are the eigenvalues of $\mathbf{H}$ for any nonzero vector $\mathbf{x}$?

Solution

starting with the definition of the householder matrix

$$\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} \tag{84}$$

$$\mathbf{H}\mathbf{v} = \left(\mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)\mathbf{v} \tag{85}$$

$$= \mathbf{v} - 2\frac{\mathbf{v}\mathbf{v}^T\mathbf{v}}{\mathbf{v}^T\mathbf{v}} \tag{86}$$

$$= -\mathbf{v} \tag{87}$$

Thus the eigenvalues of the householder matrix are all -1

Problem 2

Let $\mathbf{A}$ be $m \times n$ and $\mathbf{B}$ be $n \times m$. Show that the matrices $\begin{bmatrix} \mathbf{AB} & 0 \\ \mathbf{B} & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ \mathbf{B} & \mathbf{BA} \end{bmatrix}$ have the same eigenvalues.

Solution

First of all let us acknowlege that the dimension of $\mathbf{AB}$ must be of size $m \times m$ and the dimension of $\mathbf{BA}$ must be of size $n \times n$, therefore $\mathbf{AB} \neq \mathbf{BA}$.

Let $\mathbf{X} = \begin{bmatrix} \mathbf{AB} & 0 \\ \mathbf{B} & 0 \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} 0 & 0 \\ \mathbf{B} & \mathbf{BA} \end{bmatrix}$

We need to show that the matrices $\mathbf{X}$ and $\mathbf{Y}$ are similar, via a similarity transformation $\mathbf{S}$ as follows, i.e., $\mathbf{X} = \mathbf{S}^{-1}\mathbf{Y}\mathbf{S}$. Let us find $\mathbf{S}$.

$$\mathbf{X} = \mathbf{S}^{-1}\mathbf{Y}\mathbf{S} \tag{88}$$

$$\mathbf{S}\mathbf{X} = \mathbf{Y}\mathbf{S} \tag{89}$$

$$\begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix}\begin{bmatrix} \mathbf{AB} & 0 \\ \mathbf{B} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathbf{B} & \mathbf{BA} \end{bmatrix}\begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix} \tag{90}$$

$$\begin{bmatrix} \mathbf{EAB} + \mathbf{FB} & 0 \\ \mathbf{GAB} + \mathbf{HB} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathbf{BE} + \mathbf{BAG} & \mathbf{BF} + \mathbf{BAH} \end{bmatrix} \tag{91}$$

equating entries on both sides we have,

1. $$\mathbf{EAB} + \mathbf{FB} = \mathbf{0} \tag{92}$$

2. $$\mathbf{GAB} + \mathbf{HB} = \mathbf{BE} + \mathbf{BAG} \tag{93}$$

3. $$\mathbf{BF} + \mathbf{BAH} = \mathbf{0} \tag{94}$$

By simply doing an analysis of the compatability of the dimensions for each matrix multiplication and addition, we can see that

$$\mathbf{E} \quad \text{size} \quad m \times m \tag{95}$$
$$\mathbf{F} \quad \text{size} \quad m \times n \tag{96}$$
$$\mathbf{H} \quad \text{size} \quad n \times n \tag{97}$$
$$\mathbf{G} \quad \text{size} \quad n \times m \tag{98}$$

We choose the matrix $\mathbf{S}$ as follows:

$$\mathbf{S} = \begin{bmatrix} \mathbf{I}_m & -\mathbf{A} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix} \tag{99}$$

Let us compute the inverse $\mathbf{S}^{-1}$

Recall that in order to compute the inverse of a block matrix we have that

$$\mathbf{C}_1 = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21} \tag{100}$$
$$= \mathbf{I}_m + \mathbf{0} \tag{101}$$
$$\mathbf{C}_2 = \mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12} \tag{102}$$
$$= \mathbf{I}_n + \mathbf{0} \tag{103}$$

Thus we have that the inverse of the similarity matrix is

$$\begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{S}_{11}^{-1} + \mathbf{S}_{11}^{-1}\mathbf{S}_{12}\mathbf{C}_2^{-1}\mathbf{S}_{21}\mathbf{S}_{11}^{-1} & -\mathbf{C}_1^{-1}\mathbf{S}_{12}\mathbf{S}_{22}^{-1} \\ -\mathbf{S}_{22}^{-1}\mathbf{S}_{21}\mathbf{C}_1^{-1} & \mathbf{S}_{22}^{-1} + \mathbf{S}_{22}^{-1}\mathbf{S}_{21}\mathbf{C}_1^{-1}\mathbf{S}_{12}\mathbf{S}_{22}^{-1} \end{bmatrix} \tag{104}$$
$$= \begin{bmatrix} \mathbf{I}_m & \mathbf{I}_m\mathbf{A}\mathbf{I}_n \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix} \tag{105}$$
$$= \begin{bmatrix} \mathbf{I}_m & \mathbf{A} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix} \tag{106}$$

Let us compute $\mathbf{S}^{-1}\mathbf{Y}\mathbf{S}$

$$\mathbf{S}^{-1}\mathbf{Y}\mathbf{S} = \begin{bmatrix} \mathbf{I}_m & \mathbf{A} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{B}\mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & -\mathbf{A} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix} \tag{107}$$
$$= \begin{bmatrix} \mathbf{I}_m & \mathbf{A} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \tag{108}$$
$$= \begin{bmatrix} \mathbf{A}\mathbf{B} & \mathbf{0} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \tag{109}$$
$$= \mathbf{X} \tag{110}$$

Since $\mathbf{X}$ and $\mathbf{Y}$ are similar, they have the same characteristic polynomial, and thus share the same eigenvalues. (class notes proof 1.39)

Problem 3

Use the Gerschgorin theorem to show that the matrix

$$\mathbf{A} = \begin{bmatrix} 1.0 & 0.3 & 0.1 & 0.4 \\ 0.0 & 2.0 & 0.0 & 0.1 \\ 0.0 & 0.4 & 3.0 & 0.0 \\ 0.1 & 0.0 & 0.0 & 4.0 \end{bmatrix} \tag{111}$$

has exactly one eigenvalue in each of the four circles

$$|z - k| \leq 0.1, \quad k = 1, 2, 3, 4. \tag{112}$$

Solution

The Gerschgorin theorem states that for each diagonal entry of $\mathbf{A}$ there should be en eigenvalue located within a disk centered at the value for each diagonal entry on the argand diagram. The radius of this disk is the sum of all the non-diagonal elements in the corresponding row.

$$|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \tag{113}$$

For $i = 1$ we have a disk centered at the point $a_{11} = 1$, with radius $r_1 = 0.3 + 0.1 + 0.4 = 0.8$.

For $i = 2$ we have a disk centered at the point $a_{22} = 2$, with radius $r_2 = 0.1$.

For $i = 3$ we have a disk centered at the point $a_{33} = 3$, with radius $r_3 = 0.4$.

For $i = 4$ we have a desk centered at the point $a_{44} = 4$, with radius $r_4 = 0.1$.

Since we know that the eigenvalues of the matrix $\mathbf{A}$ and its transpose $\mathbf{A}^T$ are the same, we can apply Gerschgorin's theorem to $\mathbf{A}^T$ as well,

$$\mathbf{A}^T = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.1 \\ 0.3 & 2.0 & 0.4 & 0.0 \\ 0.1 & 0.0 & 3.0 & 0.0 \\ 0.4 & 0.1 & 0.0 & 4.0 \end{bmatrix} \tag{114}$$

For $i = 1$ we have a disk centered at the point $a^T_{11} = 1$, with radius $r_1 = 0.1$.

For $i = 2$ we have a disk centered at the point $a^T_{22} = 2$, with radius $r_2 = 0.3 + 0.4 = 0.7$.

For $i = 3$ we have a disk centered at the point $a^T_{33} = 3$, with radius $r_3 = 0.1$.

For $i = 4$ we have a desk centered at the point $a^T_{44} = 4$, with radius $r_4 = 0.1 + 0.4 = 0.5$.

thus we can bound the disk by the infimum between the two sets of disk radii, i.e., 0.1.

Additionally, since none of the circles overlap we know from the theorem that each of the disks described above much contain exactly one eigenvalue contained in the disks centered at the points $1, 2, 3, 4$ with radius 0.1.

Problem 4

Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be real and symmetric that is positive definite. Let $y \in \mathbb{R}^m$ be nonzero. Prove that the limit

$$\lim_{k \to \infty} \frac{y^T \mathbf{A}^{k+1} y}{y^T \mathbf{A}^k y} \tag{115}$$

exists and is an eigenvalue of $\mathbf{A}$.

Proof. Since $\mathbf{A}$ is a symmetric positive definite matrix, we know that

1. The eigenvalues of $\mathbf{A}$ are real.
2. $\mathbf{A}$ is diagonalizable.
3. There is an orthonormal basis of $\mathbb{R}^m$ consisting of eigenvectors of $\mathbf{A}$.

Moreover, $\mathbf{A}$ may be orthonormally diagonalized into $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^T$ where $\mathbf{V} \in \mathbb{R}^{m \times m}$ is an orthonormal matrix of eigenvectors of $\mathbf{A}$, and $\mathbf{D} \in \mathbb{R}^{m \times m}$ is a real diagonal matrix of eigenvalues corresponding to each orthogonal eigenvector.

Let $\mathbf{v} \in \mathbf{V}$ be an orthogonal eigenvector, with corresponding eigenvalue λ such that

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \tag{116}$$

We want to show that (115) exists and is an eigenvalue of $\mathbf{A}$.

Recall that under the transformation $\mathbf{A} \to \mathbf{A}^k$ the eigenvalues of $\mathbf{A}$ increase proportionally under the power, i.e.,

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \to \mathbf{A}^k \mathbf{v} = \lambda^k \mathbf{v} \tag{117}$$

Thus we can replace the $\mathbf{A}^k\mathbf{v}$ term in the limit as follows,

$$\lim_{k \to \infty} \frac{\mathbf{v}^T \mathbf{A}^{k+1} \mathbf{v}}{\mathbf{v}^T \mathbf{A}^k \mathbf{v}} = \lim_{k \to \infty} \frac{\mathbf{v}^T \lambda^{k+1} \mathbf{v}}{\mathbf{v}^T \lambda^k \mathbf{v}} \tag{118}$$

$$= \lim_{k \to \infty} \frac{\lambda^{k+1} (\mathbf{v}^T \mathbf{v})}{\lambda^k (\mathbf{v}^T \mathbf{v})} \tag{119}$$

$$= \lim_{k \to \infty} \frac{\lambda^{k+1}}{\lambda^k} \tag{120}$$

$$= \lim_{k \to \infty} \lambda = \lambda \tag{121}$$

This also holds for any non-zero vector $\mathbf{y}$ not just an eigenvector, because any non-zero vector $\mathbf{y}$ can be written as a linear combination of the orthonormal eigenvectors of $\mathbf{A}$,

$$\mathbf{y} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_m \mathbf{v}_m \tag{122}$$

where α_i are scalars and $\mathbf{v}_j$ correspond to the eigenvectors contained in $\mathbf{V}$.

Since the above result is the general case and applies to all eigenvectors, it follows that for any non-zero vector $\mathbf{y}$ the limit will converge to atleast one eigenvalue in the spectrum of $\mathbf{A}$. □

Problem 5

Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a non-defective matrix with its eigenvalues $\{\lambda_i\}_{i=1}^m$ and its singular values $\{\sigma_i\}_{i=1}^m$, satisfying

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|, \tag{123}$$
$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \tag{124}$$

Let $\rho(\mathbf{A})$ be the spectral radius of $\mathbf{A}$ and $\text{cond}(\mathbf{A}) = \|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2$ be the condition number of $\mathbf{A}$. Let $\mathbf{A}$ be normal, i.e., $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$. Show that

Part a

$$\sigma_i = |\lambda_i|, \quad 1 \leq i \leq m \tag{125}$$

Proof. Since $\mathbf{A}$ is non-defective, we know by the eigenvalue-revealing factorizations, that the diagonalization $\mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1}$ exists.

Additionally, since $\mathbf{A}$ is normal, a unitary diagonalization $\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^*$ exists as well.

Let $\sigma(\mathbf{A}) := \{\text{singular values of } \mathbf{A}\}$ and $\Lambda(\mathbf{A}) := \{\text{eigenvalues of } \mathbf{A}\}$.

By definition the singular values of the matrix $\mathbf{A}$ are the square root of the eigenvalues of the Gram-Matrix of $\mathbf{A}$.

$$\sigma(\mathbf{A}) = \sqrt{\Lambda(\mathbf{A}^*\mathbf{A})} \tag{126}$$
$$= \sqrt{\Lambda((\mathbf{QDQ}^*)^*(\mathbf{QDQ}^*))} \tag{127}$$
$$= \sqrt{\Lambda(\mathbf{QDQ}^*\mathbf{QDQ}^*)} \tag{128}$$
$$= \sqrt{\Lambda(\mathbf{QDDQ}^*)} \tag{129}$$
$$= \sqrt{\Lambda(\mathbf{QD}^2\mathbf{Q}^*)} \tag{130}$$
$$= \sqrt{\Lambda(\mathbf{D}^2)} \tag{131}$$

Thus the set of all singular values are equal to the square root of the square of the diagonal matrix $\mathbf{D}$, or equivalently

$$\sigma_i = |\lambda_i| \quad \forall i = 1, 2, 3, ..., m \tag{132}$$

□

Part b

$$\|\mathbf{A}\|_2 = |\lambda_1| = \rho(\mathbf{A}) \tag{133}$$

Proof. The 2-norm of a matrix is defined as:

$$\|\mathbf{A}\|_2 = \sqrt{\max(\text{eig}(\mathbf{A}^*\mathbf{A}))} \tag{134}$$

We have already shown above that for a normal, non-defective matrix that $\mathbf{A}^*\mathbf{A} = \mathbf{D}^2$, thus

$$\begin{aligned}
\|\mathbf{A}\|_2 &= \sqrt{\max(\text{eig}(\mathbf{D}^2))} && (135)\\
&= \sqrt{\lambda_1^2} && (136)\\
&= |\lambda_1| && (137)
\end{aligned}$$

The spectral radius of $\mathbf{A}$, by definition is the largest eigenvalue $|\lambda_1|$.

$$\|\mathbf{A}\|_2 = |\lambda_1| = \rho(\mathbf{A}) \quad (138)$$

□