Assignment 0

AM213B

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Problem 1

Plot the following functions

$$\sin(x) + \frac{x}{2} \tag{1}$$

$$\cos(x) \tag{2}$$

$$x\left(2 - \frac{x}{3} + 1\right) \tag{3}$$

in the domain $x \in [0, 7]$

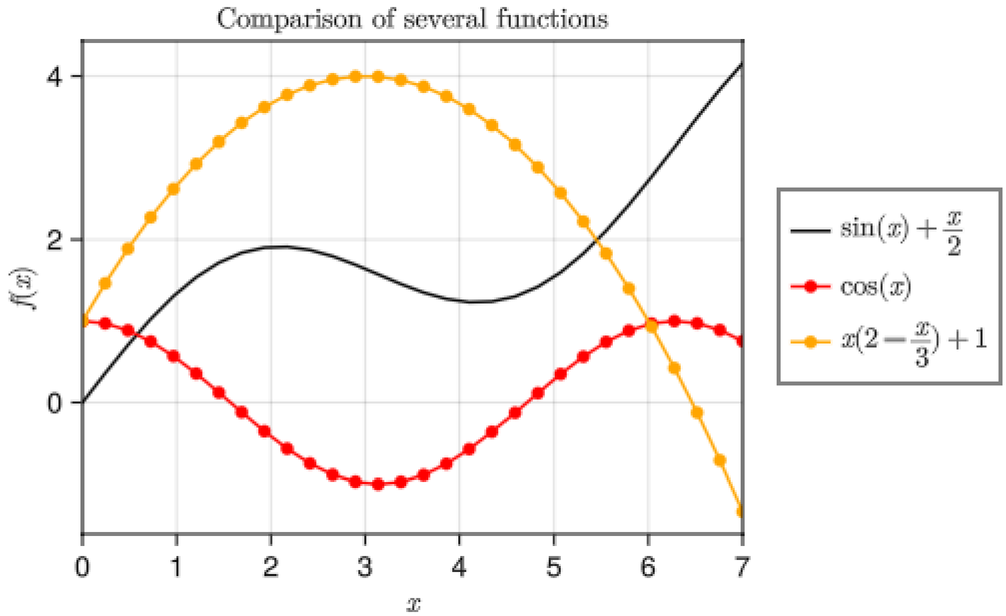
Solution

```
using GLMakie

func1(x::Float64) = sin(x) + (x*0.5)
func2(x::Float64) = cos(x)
func3(x::Float64) = x*(2 - (x/3)) + 1

function main()
    x = [i for i in LinRange(0, 7, 30)]
    data1 = [x func1.(x)]
    data2 = [x func2.(x)]
    data3 = [x func3.(x)]
    fig = Figure()
    ax = Axis(fig[1, 1], title = L"\text{Comparison of several functions}", xlabel = L"$x$", ylabel = L"$f(x)$", li
    lines!(ax, data1, color = :black, label = L"$\sin(x) + \frac{x}{2}$")
    scatterlines!(ax, data2, color = :red, label = L"$\cos(x)$")
    scatterlines!(ax, data3, color = :orange, label = L"$x(2 - \frac{x}{3}) + 1$")
    Legend(fig[1, 2], ax)

    fig
end
main()
```



Problem 2

Review the summation technique:

$$\sum_{n=0}^{N-1} r^n = \frac{r^N - 1}{r - 1} \tag{4}$$

Solution

Let $\sum_{n=0}^{N-1} r^n = S_n$ such that

$$\begin{aligned}
S_n &= 1 + r + r^2 + \cdots + r_{N-1} && (5) \\
rS_n &= r + r^2 + r^3 + \cdots + r_N && (6) \\
rS_n - S_n &= r^N - 1 && (7) \\
S_n(r-1) &= r^N - 1 && (8) \\
S_n &= \frac{r^N - 1}{r - 1} && (9) \\
\sum_{n=0}^{N-1} r^n &= \frac{r^N - 1}{r - 1} && (10)
\end{aligned}$$

Problem 3

review the technique of taylor expansion

Consider the second order numerical differentiation for approximating $f''(x)$

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + e(h) \tag{11}$$

Let us use the taylor expansion to show that $e(h) = O(h^2)$

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x) = e(h) \tag{12}$$

$$\tag{13}$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + O(h^4) \tag{14}$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + O(h^4) \tag{15}$$

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4) \tag{16}$$

$$e(h) = \frac{h^2 f''(x) + O(h^4)}{h^2} - f''(x) \tag{17}$$

$$= O(h^2) \tag{18}$$

Problem 4

Use the first and second order methods to differentiate $f(x) = \sin(x)$ at $x = 1$. In this simple problem, the exact solution is known: $f'(x) = \cos(x)$.

The exact errors of the two methods are

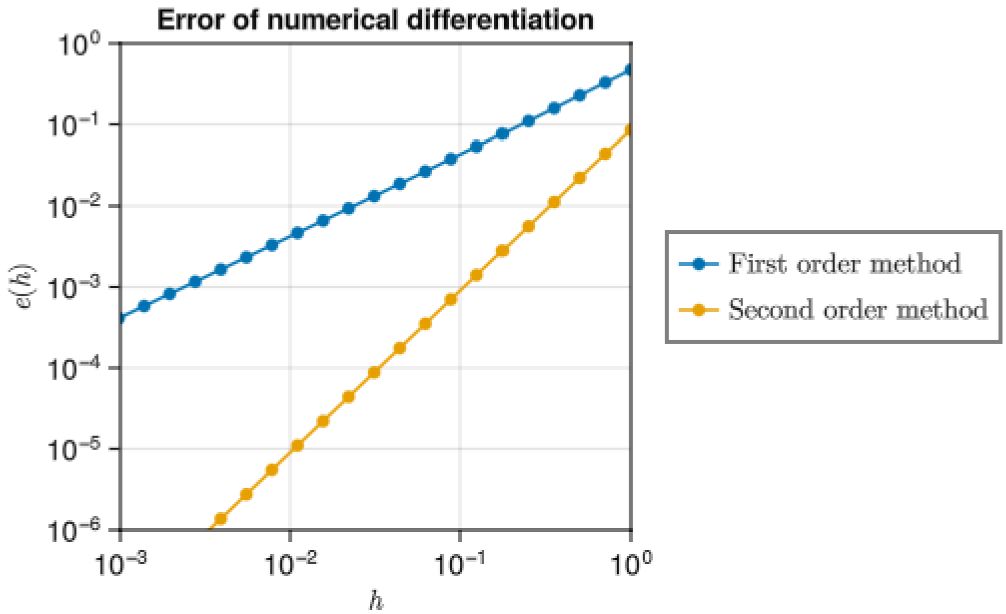
$$e_1(h) = \left|\frac{f(x+h) - f(x)}{h} - f'(x)\right|, \quad x = 1 \tag{19}$$

$$e_2(h) = \left|\frac{f(x+h) - f(x-h)}{2h} - f'(x)\right|, \quad x = 1 \tag{20}$$

Calculate $e_1(h)$ and $e_2(h)$ for $h = 2^{-i} \forall i \in \{0 : 0.5 : 10\}$. Plot h vs $e_1(h)$ and h vs $e_2(h)$ in one figure.

Solution

```
f(x) = sin(x)
fp(x) = cos(x)
er1(x, h) = abs((f(x+h) - f(x))/h - fp(x))
er2(x, h) = abs(((f(x+h) - f(x-h))/(2*h)) - fp(x))
fig = Figure()
ax = Axis(fig[1, 1], title = "Error of numerical differentiation", xlabel = L"h", ylabel = L"e(h)", xscale = log10
hs = [2^(-i) for i in 0:0.5:10]
scatterlines!(ax, hs, er1.(1.0, hs), label = L"\text{First order method}")
scatterlines!(ax, hs, er2.(1.0, hs), label = L"\text{Second order method}")
Legend(fig[1, 2], ax)
fig
```



Error of numerical differentiation
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$e(h)$
10^{-3}
10^{-2}
10^{-1}
10^0
h
First order method
Second order method