Homework 6

AM213B

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1 Problem 1

When $f(x)$ is periodic with period L, the composite trapezoidal rule for approximating

$$I = \int_0^L f(x)dx \tag{1}$$

has a very simple form

$$T(h) = \frac{h}{2}\left(f_0 + f_N + 2\sum_{i=1}^{N-1} f_i\right) = h\sum_{i=1}^{N-1} f_i \quad \text{since } \underbrace{f_0 = f_N}_{\text{periodic}} \tag{2}$$

Use the composite trapezoidal rule to calculate

$$I = \int_0^{2\pi} e^{\sin(x)} \frac{\sin(x)}{1 + \cos^2(x)} dx \tag{3}$$

Use the step size $h = \frac{2\pi}{N}, \quad N = 2^i \forall i \in [1, 9]$

Carry out the error estimation.

Use log-log to plot the estimated error $E(h)$ vs h. Also plot $\frac{h^2}{2}$ vs h in the same figure for comparison.

Remark: when applied to a periodic function, the error of the composite trapezoidal rule decreases much faster than a second order method as h is decreased.

1.1 Solution

Recall that the quantity I we want to calculate has the following relationship with the Richardson extrapolation and the numerical error estimation:

$$I = \underbrace{T(h)}_{\text{numerical approximation}} - \underbrace{E(h)}_{\text{exact error}} \tag{4}$$

We don't know the exact error so we estimate the error by

$$E(h) \approx \frac{1}{1-\frac{1}{2}} \left[T(h) - T\left(\frac{h}{2}\right) \right] \tag{5}$$

where p is the order of the method.

So the final integral value can be computed by the following:

$$I = T(h) - E(h) \approx \frac{1}{2^p - 1} \left[2^p T\left(\frac{h}{2}\right) - T(h) \right] \tag{6}$$

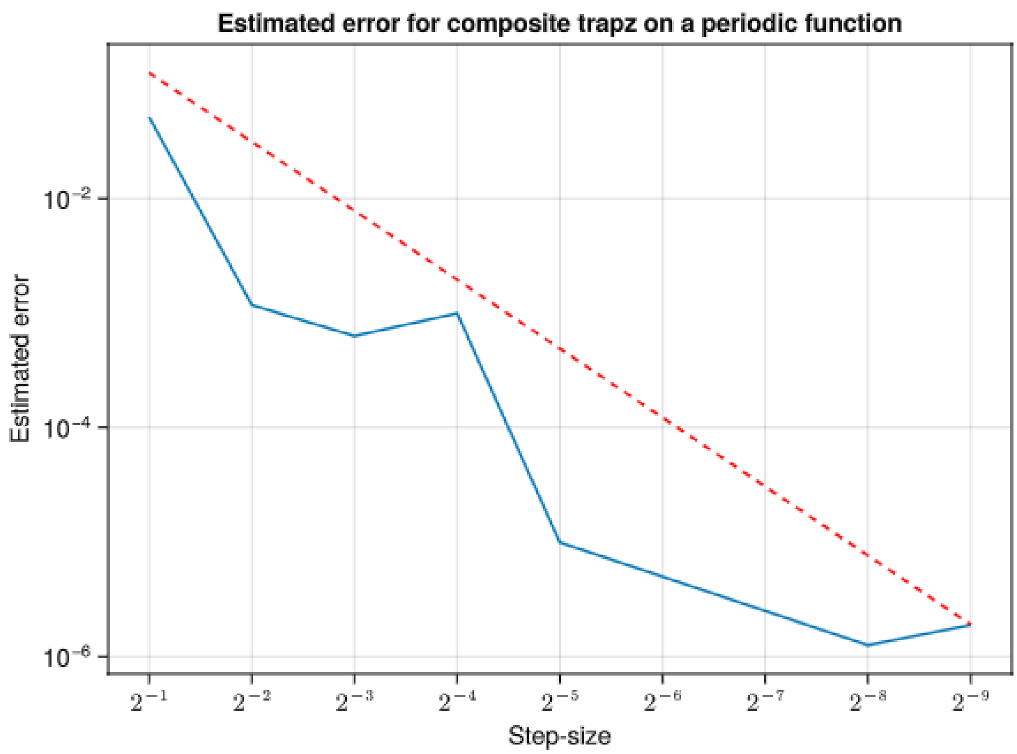

Figure 1: log-log plot of the estimated error $E(h)$ vs h in blue alongside the reference error curve for a second order method red.

2 Problem 2

Implement the FTCS method to solve,

$$\begin{cases} u_t = u_{xx} \\ u(x+2\pi, t) = u(x,t) \quad \text{periodic BC} \\ u(x,0) = (1 + 3\cos(x)) \end{cases} \tag{7}$$

Solve (7) to $T = 1.2$. Find integer N_{step} such that $\Delta t = \frac{T}{N_{\text{step}}} \approx\leq \Delta t_s$

calculate Δt and as the number of steps forward.

Let $U(x; N_x)$ be the numerical solution at time T obtained with resolution N_x.

We estimate the error in $U(x; N_x)$ numerically with

$$E(x; N_x) = U(x; N_x) - U(x; 2N_x) \tag{8}$$

When N_x is increased, both Δx and Δt decrease accordingly.

Be careful when finding the common grid points of $U(x; N_x)$ and $U(x; 2N_x)$.

When implementing the FTCS with periodic BC, we work with $\{u^n, 0 \leq i \leq (N_x + 1)\}$.

2.1 Part 1

Plot $U(x; N_x)$ vs $\frac{x}{2\pi}$ for $N_x = 100$.

2.1.1 Solution

The FTCS method (finite-time-centered-space) is as follows for (7)

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} = \alpha \frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2} \tag{9}$$

where u_j^k is an approximation of $U(x_j, t_k)$.

Thus in order to compute the next state in time we rearrange the terms to get the scheme,

$$u_j^{k+1} = u_j^k + \alpha \Delta t \frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta x^2} \tag{10}$$

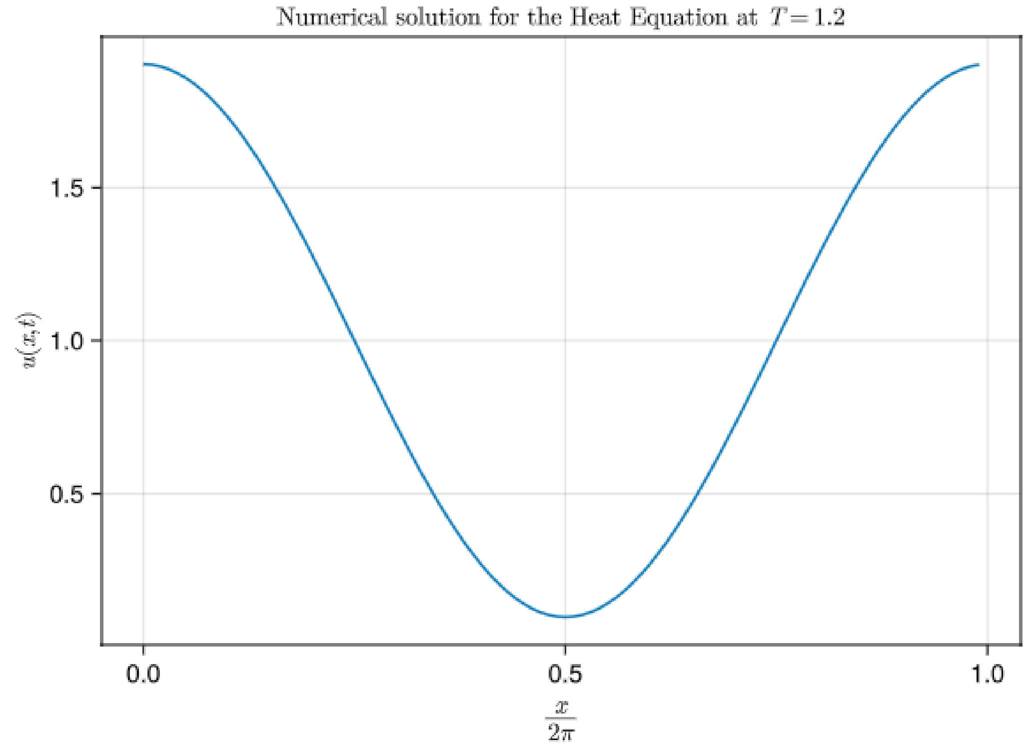

Figure 2: Heat distribution in the bar at the final time $T = 1.2$

2.2 Part 2

Plot $E(x; N_x)$ vs $\frac{x}{2\pi}$ for $N_x = 100$. Use the algebraic value of error (not the absolute value). Use linear scales for both error and x.

2.2.1 Solution

We estimate the error by using (8).

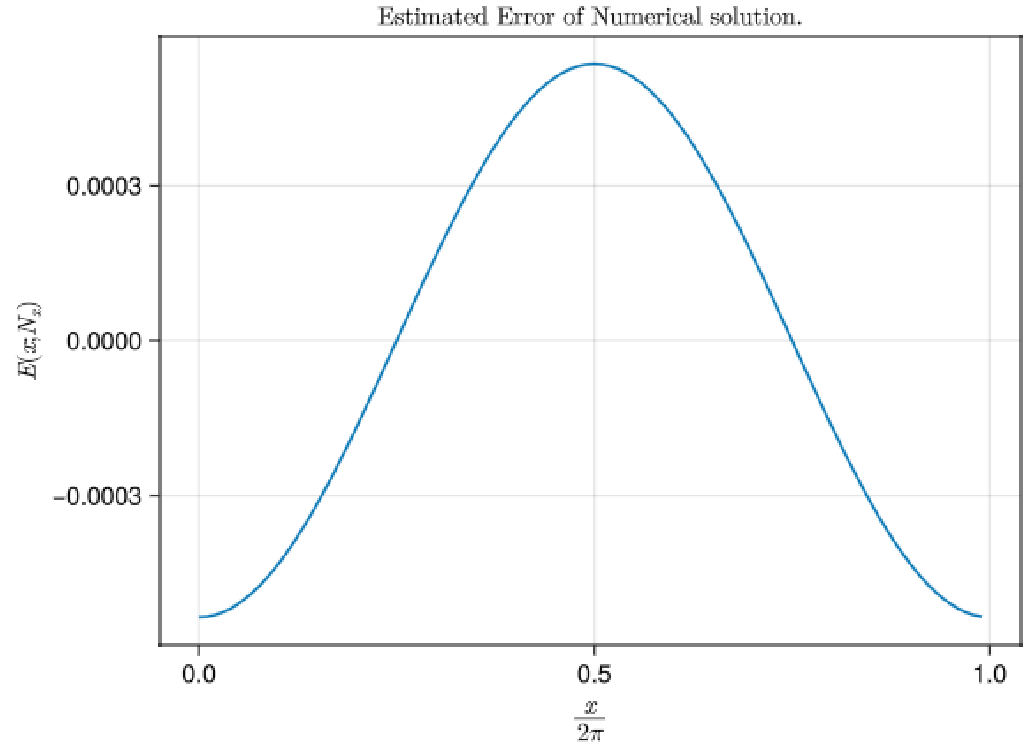

Figure 3: Error of the Numerical solution at the final time $T = 1.2$