Initial value problems for ODEs

In this lecture we briefly review the mathematical theory of initial value problems for systems of first-order ordinary differential equations (ODEs). While systems of ODEs are of great importance on their own (many real-world systems are modeled in terms of ODEs), they also play a fundamental role in the numerical approximation of PDEs.

The initial value problem for one ODE. Let us begin with the following initial value problem for just one ODE

$$\begin{cases} \dfrac{dy}{dt} = f(y,t) \\ y(0) = y_0 \end{cases} \tag{1}$$

where $f : D \times [0,T] \mapsto \mathbb{R}$ and $D \subseteq \mathbb{R}$ is a subset of $\mathbb{R}$. In order for the initial value problem (1) to be well-posed, i.e., for the problem to have a unique solution in a certain space of functions, we need to impose some mild restrictions on $f(y,t)$. As we will see, it is sufficient for f to be continuous in time and Lipschitz continuous in the domain D.

Definition 1. Let $D \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. We say that $f : D \times [0,T] \to \mathbb{R}$ is Lipschitz continuous in D if there exists a positive constant $0 \leq L < \infty$ (Lipschitz constant) such that

$$|f(y_1,t) - f(y_2,t)| \leq L\,|y_1 - y_2| \qquad \text{for all } t \in [0,T]. \tag{2}$$

The smallest number L^* such that the inequality above is satisfied is called "best" Lipschitz constant.

Lipschitz continuity is stronger than just continuity, which requires only that[1]

$$\lim_{y_1 \to y_2^{\pm}} |f(y_1,t) - f(y_2,t)| = 0 \quad \text{for all } t \in [0,T] \text{ and for all } y_2 \text{ in the interior of } D. \tag{3}$$

Indeed, Lipschitz continuity implies that the rate at which $f(y_1,t)$ approaches $f(y_2,t)$ cannot be larger than L for all y_1 and y_2 in D. In other words, a Lipschitz continuous function $f(y,t)$ has a growth rate that is bounded by L for all y_1 and y_2 in D.

Example: Let $D = [-1,1]$ be a closed interval, i.e., an interval including the endpoints -1 and 1. The function $f(y,t) = e^{-t^2} y^{1/3}$ is continuous in D for all $t \in \mathbb{R}$ (see Figure 1). However, $f(y,t)$ is not Lipschitz continuous in D. The problem here is that $f(y,t)$ has infinite "slope" at the point $y=0$ for all $t \in \mathbb{R}$. In other words, there is no constant $0 \leq L < \infty$ such that

$$|f(y,t) - f(0,t)| \leq L\,|y - 0| \quad \text{for all} \quad y \in D. \tag{4}$$

This can be seen by substituting $f(y,t) = e^{-t^2} y^{1/3}$ in (4)

$$|f(y,t)| \leq L\,|y| \quad \Rightarrow \quad e^{-t^2}\left|\frac{y^{1/3}}{y}\right| = e^{-t^2}\left|\frac{1}{y^{2/3}}\right| \leq L \quad \text{for all} \quad y \in D. \tag{5}$$

Clearly, if we send y to zero we have that L goes to infinity, and therefore $f(y,t)$ is not Lipschitz continuous in D. Note that $f(y,t)$ is Lipschitz continuous (actually infinitely differentiable with continuous derivatives), e.g., in

$$D = [-1,1] \setminus \{0\} = [-1,0[\, \cup \,]0,1] \quad \text{or in} \quad D = [1,10]. \tag{6}$$

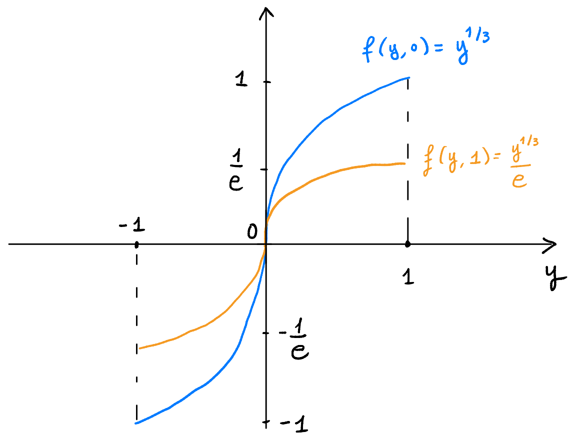


Figure 1: Sketch of $f(y,t) = e^{-t^2} y^{1/3}$ in $[-1,1]$ at $t = 0$ and $t = 1$. The function has infinite slope at $y = 0$.

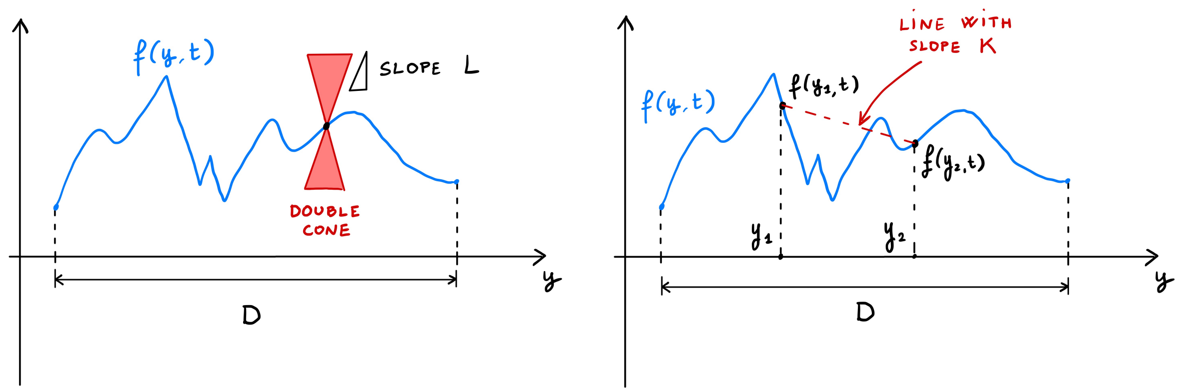


Figure 2: Geometric meaning of Lipschitz continuity.

The Lipschitz continuity condition (2) has a nice geometric interpretation. In practice it says that the function $f(y,t)$ can never enter a double cone with slope L and vertex on any point $(y, f(y,t)))$ where $y \in D$. In other words, if we can slide the vertex of the double cone over the (continuous) function $f(y,t)$ for $y \in D$ and the function never enters the cone then $f(y,t)$ is Lipschitz continuous in D. To explain this, let us divide the inequality (2) by $|y_1 - y_2|$ (for $y_1 \neq y_2$). This yields

$$\underbrace{\left| \frac{f(y_1,t) - f(y_2,t)}{y_1 - y_2} \right|}_{|K|} \leq L \quad \text{for all} \quad y_1, y_2 \in D. \tag{7}$$

For each fixed y_1 and y_2 in D we see that K represents the slope of the line connecting the points $(y_1, f(y_1,t))$ and $(y_2, f(y_2,t))$ (see Figure 2). Clearly, the best Lipshitz constant is obtained as

$$L^* = \max_{y_1, y_2 \in D} \left| \frac{f(y_1,t) - f(y_2,t)}{y_1 - y_2} \right|. \tag{8}$$

[1]The notation $y_1 \to y_2^\pm$ means that y_1 is approaching y_2 either from the left ("$-$") or from the right ("$+$"). Note that we can equivalently write (3) as

$$\lim_{y_1 \to y_2^+} f(y_1,t) = \lim_{y_1 \to y_2^-} f(y_1,t) = f(y_2,t).$$

Any $L \geq L^*$ is still a Lipschitz constant. If the function $f(y,t)$ is continuously differentiable in $y \in D$ and D is compact then

$$L^* = \max_{y\in D}\left|\frac{\partial f(y,t)}{\partial y}\right| < \infty. \tag{9}$$

Lemma 1. If $f(y,t)$ is of class C^1 in a compact subset $D \subseteq \mathbb{R}$ for all $t \in [0,T]$ then $f(y,t)$ is Lipschitz continuous in D.

Proof. By assumption the derivative of $\partial f(y,t)/\partial y$ is continuous on the compact domain $D \subseteq \mathbb{R}$. This implies that the minimum and the maximum of $\partial f(y,t)/\partial y$ is attained at some points in D. By using the mean value theorem we immediately see that

$$|f(y_1,t) - f(y_2,t)| = \left|\frac{\partial f(y^*,t)}{\partial y}\right| |y_1 - y_2| \,. \tag{10}$$

where y^* is some point within the interval $[y_1, y_2]$. The point y^* depends on f, y_1 and y_2. The right hand side of (10) can be bounded as

$$|f(y_1,t) - f(y_2,t)| \leq \underbrace{\max_{y\in D}\left|\frac{\partial f(y,t)}{\partial y}\right|}_{L^*} |y_1 - y_2| \qquad \text{for all} \quad y_1, y_2 \in D. \tag{11}$$

□

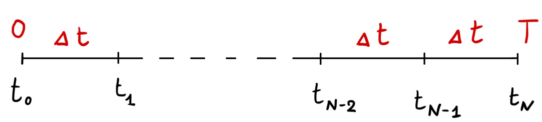
Example: The function $f(y) = y^2$ is of class C^∞ (infinitely differentiable with continuous derivative) in any bounded subset of $\mathbb{R}$. The function is not Lipshitz continuous at $y = \pm\infty$, since the slope of the first-order derivative $f'(y) = 2y$ grows unboundedly as $y \to \pm\infty$.

Remark: The initial value problem (1) can be equivalently written as

$$y(t) = y_0 + \int_0^T \frac{dy(s)}{ds} ds = y_0 + \int_0^t f(y(s),s)ds \tag{12}$$

i.e., as an integral equation for $y(s)$. This formulation is quite convenient for developing numerical methods for ODEs based on *numerical quadrature formulas*, i.e., numerical approximations of the temporal integral appearing at the right hand side of (12). For example, consider a discretization of the time interval $[0,T]$ in terms of $N+1$ evenly-spaced time instants

$$t_i = i\Delta t \qquad i = 0, 1, \ldots, N \qquad \text{where} \quad \Delta t = \frac{T}{N}. \tag{13}$$



By applying (12) within each time interval $[t_i, t_{i+1}]$ we obtain

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(y(s),s)ds. \tag{14}$$

At this point we can approximate the integral at the right hand side if (14), e.g., by using the simple rectangle rule (see Figure 3)

$$\int_{t_i}^{t_{i+1}} f(y(s),s)ds \simeq \Delta t f(y(t_i),t_i) \tag{15}$$

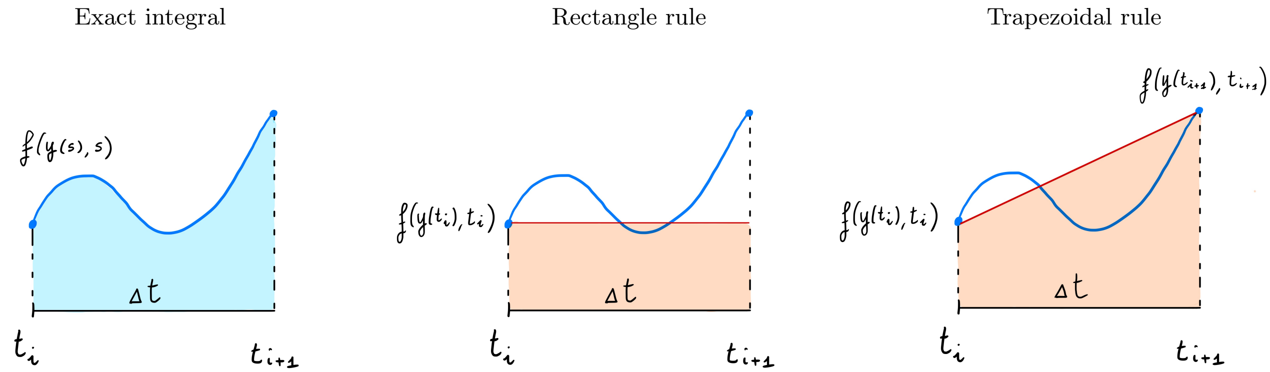


Figure 3: Approximations of the integral $\int_{t_i}^{t_{i+1}} f(y(s), s)ds$ in (14) leading to well-known numerical schemes: Euler forward (rectangle rule), Crank-Nicolson (trapezoidal rule)

This yields the *Euler forward scheme*

$$u_{i+1} = u_i + \Delta t f(u_i, t_i), \tag{16}$$

where u_i is an approximation of $y(t_i)$. The Euler forward scheme is an explicit one-step scheme. The adjective "explicit" emphasizes the fact that u_{i+1} can be computed explicitly based on the knowledge of f and u_i using (16). On the other hand, if we approximate the integral at the right hand side of (12) with the trapezoidal rule

$$\int_{t_i}^{t_{i+1}} f(y(s), s)ds \simeq \frac{\Delta t}{2}\left[f(y(t_{i+1}), t_{i+1}) + f(y(t_i), t_i)\right] \tag{17}$$

we obtain the *Crank-Nicolson scheme*

$$u_{i+1} = u_i + \frac{\Delta t}{2}\left[f(u_i, t_i) + f(u_{i+1}, t_i)\right]. \tag{18}$$

The Crank-Nicolson scheme is "implicit" because the approximate solution at time t_{i+1}, i.e., u_{i+1}, cannot be computed explicitly based on u_i, but requires the solution of a nonlinear equation. Such a solution can be computed numerically by using any method to solve nonlinear equations. These methods are usually iterative, e.g., the bisection method, or the Newton method if f is continuously differentiable. Iterative methods for nonlinear equations can be formulated as fixed point iteration problems. In the specific case of (18) we have

$$u_{i+1} = G(u_{i+1}) \qquad \text{where} \qquad G(u_{i+1}) = u_i + \frac{\Delta t}{2}\left[f(u_i, t_i) + f(u_{i+1}, t_i)\right]. \tag{19}$$

If Δt is small then u_i is close to u_{i+1}. Moreover, if Δt is sufficiently small we have that the Lipschitz constant of G is smaller than 1, which implies that the fixed point iterations will convergence globally to a unique solution u_{i+1} (see, e.g., [3, Ch. 6]).

Next, we formulate a well-known result for existence and uniqueness of the solution to the Cauchy problem for one ODE.

Theorem 1 (Well-posedness of the initial value problem for one ODE)**.** Let $D \subset \mathbb{R}$ be an open set, $y_0 \in D$. If $f : D \times [0, T] \to \mathbb{R}$ is Lipschitz continuous in D and continuous in $[0, T]$ then there exists a unique solution to the initial value problem (1) within the time interval $[0, \tau[$, where τ is the instant at which $y(t)$ exists the domain D. The solution $y(t)$ is continuously differentiable in $[0, \tau[$.

Clearly, if $f(y,t)$ is Lipschitz continuous in $y \in \mathbb{R}$ and continuous in t then the solution to the initial value problem (1) is *global* in the sense that it exists and is unique for all $t \geq 0$. This can be seen by noting that $y(t)$ never exits the domain in which $f(y,t)$ is Lipschitz continuous.

Hereafter we provide a simple example of an initial value problem that blows-up in a finite time, and an initial value problem that is not well posed.

- *Finite-time blow-up*: Consider the initial value problem

$$\frac{dy}{dt} = y^2 \qquad y(0) = 1. \tag{20}$$

  We know that $f(y) = y^2$ is non-Lipschitz at infinity. By using separation of variables it is straightforward to show that the solution to (20) is

$$y(t) = \frac{1}{1-t}. \tag{21}$$

  The function $y(t)$ clearly blows up to infinity as t approaches one (from the left).

- *Non-uniqueness of solutions:* Consider the initial value problem

$$\frac{dy}{dt} = y^{1/3} \qquad y(0) = 0. \tag{22}$$

  We have seen that $f(y) = y^{1/3}$ is not Lipshitz in any compact domain D including the point $y = 0$. In this case we are setting the initial condition exactly at the point in which the slope of $f(y)$ is infinity. By using separation of variables it can be shown that a solution to (22) is

$$y(t) = \left(\frac{2}{3}t\right)^{3/2}. \tag{23}$$

  However, as easily seen, the functions

$$y(t) = \begin{cases} 0 & \text{for } 0 \leq t < c \\ \pm\left(\frac{2}{3}(t-c)\right)^{3/2} & \text{for } t \geq c \end{cases} \tag{24}$$

  are also solutions to (22) for every $c \geq 0$.

Theorem 2 (Dependency of the ODE solution on the initial condition y_0)**.** Let $D \subset \mathbb{R}$ be an open set, $y_0 \in D$. If $f : D \times [0,T] \to \mathbb{R}$ is Lipschitz continuous in D and continuous in $[0,T]$ then the solution to (1) $y(t; y_0)$ (i.e., the flow generated by the ODE) is continuous in y_0. Moreover, if $f(y,t)$ is of class C^k in D (continuously differentiable k-times in D) then $y(t; y_0)$ is of class C^k in D.

Remark: By applying Theorem 1 iteratively (in the sense that we restart the the system from a new initial condition) we conclude that f can also be piece-wise continuous in time. This case is studied quite extensively in control of ODEs where a piecewise constant function in time is used as a control to minimize or maximize some performance metric. In this case the solution to (1) is continuous in time and piecewise diffentiable in time. The non-differentiability is at the times where the right hand side is not continuous (in t). And example of an ODE with piecewise constant control $v(t)$ is

$$\frac{dy}{dt} = g(y,t) + \underbrace{v(t)}_{\text{control}}, \qquad y(0) = y_0. \tag{25}$$

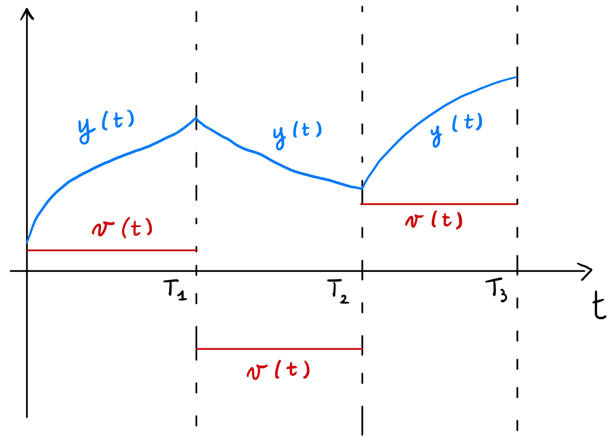


Figure 4: Piecewise differentiability of the solution in case the control $v(t)$ in equation (25) is piecewise continuous in time.

The control $v(t)$ can be computed, e.g., by solving the optimization problem

$$\min_{v(t)\in S} |y(T) - y^*|^2 \qquad \text{subject to (25)}, \tag{26}$$

where S is some function space, e.g., the space of piecewise continuous functions in $[0, T]$. Clearly, $y(T)$ depends on the whole time history of the function $v(t)$. Such functional dependence is often denoted as $y(t, [v(t)])$.

The initial value problem for systems of ODEs. Consider the following systems of nonlinear ODEs

$$\begin{cases} \dfrac{d\boldsymbol{y}}{dt} = \boldsymbol{f}(\boldsymbol{y}, t) \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases} \tag{27}$$

where $\boldsymbol{y}(t) = [y_1(t) \cdots y_n(t)]^T$ is a vector of phase variables, $\boldsymbol{f} : D \times [0, T] \to \mathbb{R}^n$, and D is a subset of $\mathbb{R}^n$. In an extended notation the system of ODEs (27) is written as

$$\begin{cases} \dfrac{dy_1}{dt} = f_1(y_1, \dots, y_n, t) \\ \dfrac{dy_2}{dt} = f_2(y_1, \dots, y_n, t) \\ \vdots \\ \dfrac{dy_n}{dt} = f_n(y_1, \dots, y_n, t) \\ y_1(0) = y_{10} \\ y_2(0) = y_{20} \\ \vdots \\ y_n(0) = y_{n0} \end{cases} \tag{28}$$

Systems of ODE such as (1) or (28) arise, e.g., when modeling physical systems (e.g., pendulum equations, UAV models, etc.) or when performing a discretization of a partial differential equation to remove dependence on spatial variables. Let us provide a simple example of a particular type of such a discretization.

Example: Consider the following initial-boundary value problem for the heat equation

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} = \alpha \dfrac{\partial^2 y(x,t)}{\partial x^2} & \text{diffusion equation} \\ y(0,x) = y_0(x) & \text{initial condition} \\ y(t,0) = y(t,2\pi) & \text{periodic boundary conditions} \end{cases} \tag{29}$$

Since this problem is defined on a periodic domain, i.e., on the circle $\mathbb{T}$, we can use a *Fourier spectral method* to discretize it in space. To this end, consider the truncated Fourier series expansion[2]

$$y_N(t,x) = \sum_{k=-N}^{N} c_k(t) e^{ikx}, \tag{32}$$

where $c_k(t)$ are time dependent functions with values in $\mathbb{C}$. The series (32) automatically satisfies the periodic boundary conditions of the problem. A substitution of (32) into (29) yields,

$$\frac{\partial y_N(t,x)}{\partial t} = \alpha \frac{\partial^2 y_N(x,t)}{\partial x^2} + \underbrace{R_N(x,t)}_{\text{residual}} \tag{33}$$

i.e.,

$$\sum_{k=-N}^{N} \frac{dc_k(t)}{dt} e^{ikx} = -\alpha \sum_{k=-N}^{N} k^2 c_k(t) e^{ikx} + R_N(x,t) \tag{34}$$

At this point we impose that the residual PDE $R_N(x,t)$ is orthogonal o the span of the basis $B_N = \{e^{ikx}\}_{k=-N}^{N}$ in the sense of the standard inner product

$$(u,v)_{L^2([0,2\pi]} = \int_0^{2\pi} u(x)v(x)dx, \tag{35}$$

i.e.,

$$\left(R_N(x,t), e^{-ijx}\right)_{L^2([0,2\pi]} = 0 \qquad j = -N, ..., N. \tag{36}$$

This is called Fourier-Galerkin method [2, p.43], and yields a linear systems of $2N+1$ ODEs for the Fourier coefficients c_k

$$\frac{dc_k(t)}{dt} = -\alpha k^2 c_k(t), \qquad c_k(0) = \frac{1}{2\pi}\int_0^{2\pi} y_0(x) e^{-ikx} dx \qquad k = -N, ..., N. \tag{37}$$

Note that this system can be solved analytically. The solution is as

$$c_k(t) = \frac{e^{-\alpha k^2 t}}{2\pi} \int_0^{2\pi} y_0(x) e^{-ikx} dx \tag{38}$$

[2] The convergence rate of the Fourier series (32) to $y(x,t)$ depends on the smoothness of $y(x,t)$ in $x \in [0,2\pi]$. Specifically, it can be shown that if $y(x,t) \in H^q([0,2\pi])$ (Sobolev space with degree q) for all t then [2, p. 35]

$$\|y(x,t) - y_N(x,t)\|^2_{L^2([0,2\pi])} \leq CN^{-q} \left\| \frac{d^q y}{dx^q} \right\|^2_{L^2([0,2\pi])}. \tag{30}$$

This type of convergence is called *spectral converge*. Moreover, if $y(x,t)$ is analytic in x for all t then it can be shown that

$$\|y(x,t) - y_N(x,t)\|^2_{L^2([0,2\pi])} \leq Qe^{-cN} \|y\|^2_{L^2([0,2\pi])}, \tag{31}$$

i.e., convergence is *exponential* [2, p. 36].

which allows yields the approximate solution

$$y_N(x,t) = \frac{1}{2\pi}\sum_{k=-N}^{N} e^{ikx-\alpha k^2 t}\int_0^{2\pi} y_0(x)e^{-ikx}dx. \tag{39}$$

As we will see, the IBVP (29) can be discretized in space by using many other techniques including finite-difference methods, pseudo-spectral collocation methods, finite-elements methods, etc. The proper way to formulate these methods often goes through the so-called weak (or variational) form of the PDE. AM 213B focuses mostly on finite-difference approximation methods of PDEs. For example, a central finite-difference approximation of the PDE (29) yields the ODE system

$$\frac{du(x_k,t)}{dt} = \frac{\alpha}{\Delta x^2}\left(u(x_{k+1},t) - 2u(x_k,t) + u(x_{k-1},t)\right) \qquad u(x_{N+j},t) = u(x_j,t) \tag{40}$$

where

$$x_k = k\Delta x \qquad k = 0,\ldots,N, \qquad \Delta x = \frac{2\pi}{(N+1)} \quad \text{(uniform grid spacing)}. \tag{41}$$

Clearly, this system can be written in the form (28) provided we define

$$y_k(t) = u(x_k,t) \qquad f_k(y_1,\ldots,y_n) = \frac{\alpha}{\Delta x^2}\left(y_{k+1} - 2y_k + y_{k-1}\right) \tag{42}$$

As before, we can re-write the Cauchy problem as an integral equation

$$\boldsymbol{y}(t) = \boldsymbol{y}(0) + \int_0^t \boldsymbol{f}(\boldsymbol{y}(s),s)ds, \tag{43}$$

which is very handy to derive numerical methods based on numerical quadrature of the one-dimensional integral at the right hand side. For instance, consider a partition of the $[0,T]$ into an evenly spaced grid points such that $t_{i+1} = t_i + \Delta t$, and write (43) within each time interval

$$\boldsymbol{y}(t_{i+1}) = \boldsymbol{y}(t_i) + \int_{t_i}^{t_{i+1}} \boldsymbol{f}(\boldsymbol{y}(s),s)ds. \tag{44}$$

By approximating the integral at the right hand side of (44), e.g., using the midpoint rule yields

$$\int_{t_i}^{t_{i+1}} \boldsymbol{f}(\boldsymbol{y}(s),s)ds \simeq \Delta t \boldsymbol{f}\left(\boldsymbol{y}\left(t_i + \frac{\Delta t}{2}\right), t_i + \frac{\Delta t}{2}\right) \tag{45}$$

At this point, we can approximate $\boldsymbol{y}(t_i + \Delta t/2)$ using the Euler forward method

$$\boldsymbol{y}\left(t_i + \frac{\Delta t}{2}\right) \simeq \boldsymbol{y}(t_i) + \frac{\Delta t}{2}\boldsymbol{f}(\boldsymbol{y}(t_i),t_i) \tag{46}$$

to obtain the *explicit midpoint method*

$$\boldsymbol{u}_{i+1} = \boldsymbol{u}_i + \Delta t \boldsymbol{f}\left(\boldsymbol{u}_i + \frac{\Delta t}{2}\boldsymbol{f}(\boldsymbol{u}_i,t_i), t_i + \frac{\Delta t}{2}\right) \tag{47}$$

where $\boldsymbol{u}_i$ is an approximation of $\boldsymbol{f}(t_i)$. The explicit midpoint method is a one-step method that belongs to the class of Runge-Kutta methods[3]. The integral formulation (43) is also at the basis of the Picard iteration method which is used to prove the following theorem.

[3] As we will see, the explicit midpoint method (47) is a two-stage explicit Runge-Kutta method.

Theorem 3 (Well-posedness of initial value problems for systems of ODEs)**.** Let $D \subset \mathbb{R}^n$ be an open set, $\boldsymbol{y}_0 \in D$. If $\boldsymbol{f} : D \times [0, T] \to \mathbb{R}$ is Lipschitz continuous in D and continuous in $[0, T]$ then there exists a unique solution to the initial value problem (27) within the time interval $[0, \tau[$, where τ is defined to be the instant at which $\boldsymbol{y}(t)$ exits the domain D in which $\boldsymbol{f}$ is Lipschitz continuous. The solution $\boldsymbol{y}(t)$ is continuously differentiable in $[0, \tau[$.

How do we define Lipschitz continuity for a vector-valued function $\boldsymbol{f}(\boldsymbol{y}, t)$ defined in subset of $\mathbb{R}^d$? By a simple generalization of the definition we gave for one-dimensional functions.

Definition 2. Let D be a subset of $\mathbb{R}^n$, $\boldsymbol{f} : D \times [0, T] \to \mathbb{R}^n$. We say that $\boldsymbol{f}$ is Lipschitz continuous in D if there exists a constant $0 \leq L < \infty$ such that

$$\|\boldsymbol{f}(\boldsymbol{y}_1, t) - \boldsymbol{f}(\boldsymbol{y}_2, t)\| \leq L \|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \quad \text{for all} \quad \boldsymbol{y}_1, \boldsymbol{y}_2 \in D, \tag{48}$$

where $\|\cdot\|$ is any norm defined in $\mathbb{R}^n$.

Remark: As is well known, all norms defined in a finite-dimensional vector space (such as $\mathbb{R}^n$) are *equivalent*. This means that if we pick two arbitrary norms in $\mathbb{R}^n$, say $\|\cdot\|_a$ and $\|\cdot\|_b$, then there exist two numbers c_1 and c_2 such that

$$c_1 \|\boldsymbol{y}\|_a \leq \|\boldsymbol{y}\|_b \leq c_2 \|\boldsymbol{y}\|_a \quad \text{for all} \quad \boldsymbol{y} \in \mathbb{R}^n. \tag{49}$$

The most common norms in $\mathbb{R}^n$ are

$$\|\boldsymbol{y}\|_\infty = \max_{k=1,..,n} |y_k| , \tag{50}$$

$$\|\boldsymbol{y}\|_1 = \sum_{k=1}^{n} |y_k| , \tag{51}$$

$$\|\boldsymbol{y}\|_2 = \left(\sum_{k=1}^{n} |y_k|^2\right)^{1/2} , \tag{52}$$

$$\vdots \tag{53}$$

$$\|\boldsymbol{y}\|_p = \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p} \qquad p \in \mathbb{N} \setminus \{\infty\}. \tag{54}$$

Based on these definitions it is easy to show, e.g., that

$$\|\boldsymbol{y}\|_\infty \leq \|\boldsymbol{y}\|_1 \leq n \|\boldsymbol{y}\|_\infty , \tag{55}$$

$$\|\boldsymbol{y}\|_2 \leq \|\boldsymbol{y}\|_1 \leq \sqrt{n} \|\boldsymbol{y}\|_2 , \tag{56}$$

$$\|\boldsymbol{y}\|_\infty \leq \|\boldsymbol{y}\|_2 \leq \sqrt{n} \|\boldsymbol{y}\|_\infty . \tag{57}$$

Therefore if the $\boldsymbol{f}(\boldsymbol{y}, t)$ is Lipschitz continuous in D with respect to the 1-norm, i.e.,

$$\|\boldsymbol{f}(\boldsymbol{y}_1, t) - \boldsymbol{f}(\boldsymbol{y}_2, t)\|_1 \leq L_1 \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_1 \quad \text{for all} \quad \boldsymbol{y}_1, \boldsymbol{y}_2 \in D, \quad \text{for all} \quad t \geq 0 \tag{58}$$

then it is also Lipschitz continuous with respect to the uniform norm. In fact, by using (55) we have

$$\|\boldsymbol{f}(\boldsymbol{y}_1, t) - \boldsymbol{f}(\boldsymbol{y}_2, t)\|_\infty \leq \underbrace{L_1 n}_{L_\infty} \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_\infty . \tag{59}$$

Of course, $\boldsymbol{f}(\boldsymbol{y}, t)$ is also Lipschitz continuous with respect to the 2-norm.

Theorem 4. If $\boldsymbol{f}(\boldsymbol{y},t)$ is of class C^1 in a compact convex domain $D \subset \mathbb{R}^n$, then $\boldsymbol{f}(\boldsymbol{y},t)$ is Lipschitz continuous in D.

Proof. Let $D \subseteq \mathbb{R}^n$ be a compact convex domain and let

$$M = \max_{\boldsymbol{y}\in D} \left| \frac{\partial f_j(\boldsymbol{y},t)}{\partial y_i} \right|. \tag{60}$$

Clearly M exists and is finite because we assumed that D is compact and that $\boldsymbol{f}$ is of class C^1 in D[4]. Consider two points $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ in D, and the line that connects $\boldsymbol{y}_1$ to $\boldsymbol{y}_2$, i.e.,

$$\boldsymbol{z}(s) = (1-s)\boldsymbol{y}_1 + s\boldsymbol{y}_2 \qquad s \in [0,1]. \tag{61}$$

Since D is convex, we have that the line $\boldsymbol{z}(s)$ lies entirely within D. Therefore we can use the mean value theorem applied to the one-dimensional function $f_i(\boldsymbol{z}(s),t)$ $(s \in [0,1])$ to obtain

$$f_i(\boldsymbol{y}_2,t) - f_i(\boldsymbol{y}_1,t) = \nabla f_i(\boldsymbol{z}(s^*),t)\cdot(\boldsymbol{y}_2 - \boldsymbol{y}_1) \qquad \text{for some } s^* \in [0,1]. \tag{62}$$

By taking the absolute value and using the Cauchy-Schwartz inequality we obtain

$$\begin{aligned} |f_i(\boldsymbol{y}_2,t) - f_i(\boldsymbol{y}_1,t)|^2 &= \left| \sum_{j=1}^n \frac{\partial f_i(\boldsymbol{z}(s^*))}{\partial y_j}(y_{2j} - y_{1j}) \right|^2 \\ &\le \left| \sum_{j=1}^n \frac{\partial f_i(\boldsymbol{z}(s^*),t)}{\partial y_j} \right|^2 \left| \sum_{j=1}^n (y_{2j} - y_{1j}) \right|^2 \\ &\le nM^2 \left\| \boldsymbol{y}_2 - \boldsymbol{y}_1 \right\|_2^2. \end{aligned} \tag{63}$$

This implies that

$$\left\| \boldsymbol{f}(\boldsymbol{y}_2,t) - \boldsymbol{f}(\boldsymbol{y}_1,t) \right\|_2 \le \underbrace{nM}_{L_2} \left\| \boldsymbol{y}_2 - \boldsymbol{y}_1 \right\|_2. \tag{64}$$

i.e., $\boldsymbol{f}(\boldsymbol{y},t)$ is Lipschitz continuous in the 2-norm, or any other norm that is equivalent to the 2-norm. In particular, by using the inequalities (55)-(57) we have that $\boldsymbol{f}(\boldsymbol{y},t)$ is Lipschitz continuous relative to the 1-norm and the uniform norm (∞-norm).

□

Lemma 2. If $\boldsymbol{f}(\boldsymbol{y},t)$ is of class C^1 in $D \subseteq \mathbb{R}^n$ and has bounded derivatives $\partial f_i/\partial y_j$ then $\boldsymbol{f}(\boldsymbol{y},t)$ is Lipschitz continuous in D.

Linear systems of ODEs. Consider the following autonomous system of linear differential equations

$$\frac{d\boldsymbol{y}(t)}{dt} = \boldsymbol{A}\boldsymbol{y}(t) \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0. \tag{65}$$

We have seen in AM214 that this system admits a global solution, i.e., the solution exists and is unique for all $t \ge 0$. Such an analytic solution can be expressed in terms of generalized eigenvectors of $\boldsymbol{A}$ as

$$\begin{aligned} \boldsymbol{y}(t) &= e^{t\boldsymbol{A}}\boldsymbol{y}_0 \\ &= \boldsymbol{P}e^{t\boldsymbol{J}}\boldsymbol{P}^{-1}\boldsymbol{y}_0, \end{aligned} \tag{66}$$

[4] A compact domain is by definition bounded and closed. The minimum and maximum of a continuous function in defined on a compact domain is attained at some points within the domain or on its boundary. Note that this is not true if the domain is not compact. For example, the function $f(y) = 1/y$ is continuously differentiable on $]0,1]$ (bounded domain by not compact), but the function is unbounded on $]0,1]$.

where $\boldsymbol{P}$ is a matrix that has the generalized eigenvectors of $\boldsymbol{A}$ as columns, and $\boldsymbol{J}$ is the Jordan form of $\boldsymbol{A}$. While the formula (66) is nice and compact, its computation requires the knowledge of the eigenvalues and and generalized eigenvectors of $\boldsymbol{A}$ which is something that is not easy to compute, especially in high-dimensions[5]. Moreover, the matrix $\boldsymbol{A}$ can be time-dependent (i.e., $\boldsymbol{A}(t)$), in which case the matrix exponential $e^{t\boldsymbol{A}}$ has to be replaced by a Magnus series (see, e.g., [1]).

Matrix norms compatible with vector norms Let us define the following matrix norm

$$\|\boldsymbol{A}\| = \sup_{\boldsymbol{y}\neq \boldsymbol{0}_{\mathbb{R}^n}} \frac{\|\boldsymbol{A}\boldsymbol{y}\|}{\|\boldsymbol{y}\|} = \sup_{\|\boldsymbol{y}\|=1} \|\boldsymbol{A}\boldsymbol{y}\|. \tag{67}$$

Clearly, $\|\boldsymbol{A}\|$ is matrix norm (prove it as exercise), which satisfies, by definition, the following inequality

$$\|\boldsymbol{A}\| \geq \frac{\|\boldsymbol{A}\boldsymbol{y}\|}{\|\boldsymbol{y}\|} \qquad \text{i.e.} \qquad \|\boldsymbol{A}\boldsymbol{y}\| \leq \|\boldsymbol{A}\| \, \|\boldsymbol{y}\|. \tag{68}$$

It is straightforward to show that

$$\|\boldsymbol{A}\|_\infty = \max_{i=1,..,n} \left(\sum_{j=1}^{n} |A_{ij}| \right), \tag{69}$$

$$\|\boldsymbol{A}\|_1 = \max_{j=1,..,n} \left(\sum_{i=1}^{n} |A_{ij}| \right), \tag{70}$$

$$\|\boldsymbol{A}\|_2 = \sqrt{\lambda_{\max}\left(\boldsymbol{A}^T\boldsymbol{A}\right)} = \sigma_{\max}(\boldsymbol{A}), \tag{71}$$

where $\sigma_{\max}(\boldsymbol{A})$ is the largest singular value of the matrix $\boldsymbol{A}$. For example,

$$\|\boldsymbol{A}\boldsymbol{y}\|_\infty = \max_{i=1,\dots,n} \left| \sum_{j=1}^{n} A_{ij} y_j \right| \leq \max_{i=1,\dots,n} \left(\sum_{j=1}^{n} |A_{ij}| \, |y_j| \right) \leq \|\boldsymbol{y}\|_\infty \max_{i=1,\dots,n} \left(\sum_{j=1}^{n} |A_{ij}| \right) \tag{72}$$

which implies that

$$\frac{\|\boldsymbol{A}\boldsymbol{y}\|_\infty}{\|\boldsymbol{y}\|_\infty} \leq \max_{i=1,\dots,n} \left(\sum_{j=1}^{n} |A_{ij}| \right) \qquad \text{for all } \boldsymbol{y} \neq \boldsymbol{0}_{\mathbb{R}^n}, \tag{73}$$

i.e.,

$$\sup_{\boldsymbol{y}\neq \boldsymbol{0}_{\mathbb{R}^n}} \frac{\|\boldsymbol{A}\boldsymbol{y}\|_\infty}{\|\boldsymbol{y}\|_\infty} = \max_{i=1,\dots,n} \left(\sum_{j=1}^{n} |A_{ij}| \right) = \|\boldsymbol{A}\|_\infty. \tag{74}$$

With any compatible matrix norm available we immediately see that the function $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{y}$ is Lipschitz continuous in $\mathbb{R}^n$. In fact, we have

$$\|\boldsymbol{A}\boldsymbol{y}_1 - \boldsymbol{A}\boldsymbol{y}_2\| \leq \|\boldsymbol{A}\| \, \|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \qquad \text{for all} \qquad \boldsymbol{y}_1, y_2 \in \mathbb{R}^n, \tag{75}$$

where $L = \|\boldsymbol{A}\|$ is the Lipschitz constant. Equation (75) implies that the solution to (65) is global in time. This can be also shown by noticing that $\boldsymbol{A}$ is the Jacobian matrix of $\boldsymbol{f}(\boldsymbol{y}, t)$ and that all entries of such a matrix are of course bounded in $\mathbb{R}^n$ (see Lemma 2).

[5] If the matrix $\boldsymbol{A}$ has a particular structure, e.g., if $\boldsymbol{A}$ is a tridiagonal differentiation matrix (Toeplitz matrix), then there are formulas available for the eigenvalues and the eigenvectors of $\boldsymbol{A}$.

The following result on the regularity of the flow generated by the initial value problem (27) holds true.

Theorem 5 (Dependency of the ODE solution on the initial condition $\boldsymbol{y}_0$)**.** Let $D \subset \mathbb{R}^n$ be an open set, $\boldsymbol{y} \in D$. If $\boldsymbol{f} : D \times [0, T] \to \mathbb{R}^n$ is Lipschitz continuous in D and continuous in $[0, T]$ then the solution $\boldsymbol{y}(t; \boldsymbol{y}_0)$ to the initial value problem (27), i.e., flow generated by the ODE system, is continuous in $\boldsymbol{y}_0$. Moreover, if $\boldsymbol{f}(\boldsymbol{y}, t)$ is of class C^k (continuously differentiable k-times in D) in D then $\boldsymbol{y}(t; \boldsymbol{y}_0)$ is of class C^k in D relative to $\boldsymbol{y}_0$.

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